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Matrix approach to Lagrangian fluid dynamics

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A new approach to ideal-fluid hydrodynamics based on the notion of continuous deformation of infinitesimal material elements is proposed. The matrix approach adheres to the Lagrangian (material) view of fluid motion, but instead of Lagrangian particle trajectories, it treats the Jacobi matrix of their derivatives with respect to Lagrangian variables as the fundamental quantity completely describing fluid motion.

A closed set of governing matrix equations equivalent to conventional Lagrangian equations is formulated in terms of this Jacobi matrix. The equation of motion is transformed into a nonlinear matrix differential equation in time only, where derivatives with respect to the Lagrangian variables do not appear. The continuity equation that requires constancy of the Jacobi determinant in time takes the form of an algebraic constraint on the Jacobi matrix. An accompanying linear consistency condition, which is responsible for the dependence on spatial variables and does not include time derivatives, ensures completeness of the system and reconstruction of the particle trajectories by the Jacobi matrix.

A class of exact solutions to the matrix equations that describes rotational non-stationary three-dimensional motions having no analogues in the conventional formulations is also found and investigated. A distinctive feature of these motions is precession of vortex lines (rectilinear or curvilinear) around a fixed axis in space. Boundary problems for the derived exact solutions including matching of rotational and potential motions across the boundary of a vortex tube are addressed. In particular, for the cylindrical vortex of elliptical cross-section involved in three-dimensional precession, the outer potential flow is constructed and shown to be a non-stationary periodic straining flow at a large distance from the vortex axis.

1. Introduction

There exist several formulations of hydrodynamic equations for studying various types of fluid motions: either fluid velocity is assumed to be a function of coordinates, or coordinates of fluid particles are considered to be functions of their initial values, or complex potential or velocity are sometimes taken as independent variables, and so on. Choosing an adequate approach may help one simplify the mathematical model even of very complicated phenomena. Therefore, for an efficient approach to three-dimensional fluid dynamics problems it is interesting to revise the existing theories.

The conventional Eulerian approach is most widely used in classical and modern fluid dynamics. Within the framework of this approach the velocity field or other related characteristics of motion are considered to be functions of spatial coordinates and time. An alternative Lagrangian representation is focused on observing the motion of individual fluid particles identified by three parameters known as Lagrangian variables. The Lagrangian formulation was employed in a number of classical papers (see Lamb 1932; Kochin, Kibel & Roze 1964) but is seldom used in modern fluid dynamics. The probable reason for this is that viscous terms in the Lagrangian equations take an extremely complex form (see Monin & Yaglom 1975) making it difficult to account for viscous effects. However, the Lagrangian method offers certain advantages in an ideal-fluid theory. For instance, one can make use of conservation of any Lagrangian invariant (i.e. the quantity that remains constant within a small fluid mass in the course of motion) to reduce the order of governing equations, since each Lagrangian invariant represents their explicit time-independent integral.

In this paper we propose a novel approach to ideal-fluid hydrodynamics.† It rests essentially on the Lagrangian (material) representation of fluid motion but differs from the conventional Lagrangian formulation as follows. Fluid motion is considered as continuous deformation of infinitesimal material elements $\mathrm{d}\boldsymbol{X}$ from their initial states at $t = 0$, $\mathrm{d}\boldsymbol{X}_0$, to the current ones at $t > 0$ (a material element implies a small vector between two close individual fluid particles). Evolution of the differential element $\mathrm{d}\boldsymbol{X}$ can be expressed as $\mathrm{d}\boldsymbol{X} = \boldsymbol{R}\, \mathrm{d}\boldsymbol{X}_0$, where the matrix $\boldsymbol{R} = (\partial X_i/\partial X_{0k})$ describes stretching and turning of the vector. Knowledge of matrix $\boldsymbol{R}$ as a function of spatial coordinates and time provides a complete description of the flow. A closed set of equations for this matrix equivalent to conventional ideal-fluid equations is formulated below. Within the framework of this approach matrix $\boldsymbol{R}$ plays the role of the principal characteristic of motion like the velocity vector in the Eulerian theory or a particle trajectory in the classical Lagrangian formulation.

Deformation of material elements is a non-trivial matter of considerable physical interest even in its kinematical aspect, i.e. for a prescribed Eulerian velocity field (see Dresselhaus & Tabor 1991). The matrix approach which is intrinsically bound up with deformation of material elements allows one to obtain immediately a complete description of their kinematics from the solution of the governing matrix equations.

Using a tensor description in fluid mechanics seems quite natural since hydrodynamics is a particular case of the dynamics of a deformable continuum. But the tensor quantities of common use such as strain tensor or strain rate tensor are unsuitable for formulating a closed set of tensor equations. The inadequacy of both the tensors follows from the fact that they cannot describe directly the rotational part of velocity, which is essential in fluid dynamics. Unlike the tensors mentioned, the Lagrangian tensor $\boldsymbol{R}$ allows one to account for the rotational component of motion. However, a closed system of tensor equations for $\boldsymbol{R}$ governing the motion of an ideal fluid seems to have never been formulated. This fact does not mean that matrix $\boldsymbol{R}$ has never been applied in fluid mechanics. For instance, it appears in the formula for vorticity

$$\boldsymbol{\Omega} = \boldsymbol{R}\, \boldsymbol{\Omega}_0, \quad \boldsymbol{\Omega}_0 = \boldsymbol{\Omega}|_{t=0}$$

known as the Cauchy equation (see Batchelor 1967; Saffman 1992, § 1.7). This equation played an important role in numerous studies of turbulence in large-scale flows within the framework of the rapid distortion theory, reviewed by Hunt & Carruthers (1990).

The matrix $\boldsymbol{R}$ was also introduced as an auxiliary quantity for succinct notation in the case of linear dependence of particle coordinates on the Lagrangian variables when elements of matrix $\boldsymbol{R}$ are functions of time only. A detailed review of recent results on this type of motion with a free surface together with problems of their stability can be found in Andreev (1992). However, the general formulation of governing equations that is primarily employed in that book involves not only matrix $\boldsymbol{R}$ but

† Some elements of the proposed approach were reported in Abrashkin, Zenkovich & Yakubovich (1997).

also velocity vector and pressure. Moreover, only a particular form of these equations that is limited to linear coordinate dependence is then studied. Thus, the equations found in Andreev (1992) cannot be considered as an adequate closed matrix form of fluid dynamics equations.

The new matrix formulation of governing equations is discussed in general in § 2. Matrix techniques are then developed in § 3 in order to derive and analyse particular exact solutions. Several classes of solutions describing three-dimensional non-stationary rotational motions are found and studied in detail.

It is of interest whether the analytical solutions obtained can be applied to localized vortices in a potential flow. The main difficulty in developing such a model is matching the rotational flow and the potential one with necessary continuity conditions satisfied across the boundary of the vortex. It should be noted that analytical matching is generally problematic in every known approach to vortex dynamics. In § 4 we perform matching for a particular case of the derived exact solutions and construct a model of a precessing cylindrical vortex of elliptical cross-section in an external three-dimensional irrotational strain.

2. Matrix form of the hydrodynamic equations

2.1. *Conventional Lagrangian approach*

The Lagrangian representation of fluid motion is based on the idea of fluid particles or material points which are infinitesimal volumes of fluid that remain as individual entities in the course of motion along their trajectories. Fluid particles are distinguished by three parameters a, b, c known as Lagrangian variables which must be in one-to-one correlation with the initial positions of particles, $\boldsymbol{X}_0$. The motion is described by the set of trajectories $\boldsymbol{X}(a, b, c, t)$, where $\boldsymbol{X} = \{X, Y, Z\}$, X, Y, Z are Cartesian coordinates. The current position of particles, $\boldsymbol{X}$, is considered as a function of time t and the Lagrangian variables a, b, c.

The system of Lagrangian governing equations for an ideal incompressible fluid comprises two equations: the equation of motion and the equation of continuity (see Lamb 1932). The equation of motion follows from Newton's second law for a fluid particle

$$\boldsymbol{X}_{tt} = -\frac{1}{\rho}\nabla p - \nabla \Phi, \tag{2.1}$$

where p is pressure, ρ is density (hereafter assumed constant and uniform), and $\nabla\Phi$ is the external potential force per unit mass. The gradients in the right-hand side of (2.1) are taken with respect to Cartesian space coordinates X, Y, Z, but in the Lagrangian representation the coordinates play the role of unknown functions of a, b, c. Therefore, (2.1) is converted into a system that describes the dependence of the trajectories on the Lagrangian variables in an explicit form:

$$\left.\begin{aligned} X_{tt}X_a + Y_{tt}Y_a + Z_{tt}Z_a &= -p_a/\rho - \Phi_a, \\ X_{tt}X_b + Y_{tt}Y_b + Z_{tt}Z_b &= -p_b/\rho - \Phi_b, \\ X_{tt}X_c + Y_{tt}Y_c + Z_{tt}Z_c &= -p_c/\rho - \Phi_c. \end{aligned}\right\} \tag{2.2}$$

Hereafter, the subscripts a, b, c, t denote partial differentiation with respect to the specified variable.

For an incompressible fluid the continuity equation sets the invariance condition on the volume of a Lagrangian particle, that is the Jacobian of X, Y, Z with respect to a, b, c must be independent of time:

$$\frac{\mathrm{D}(X, Y, Z)}{\mathrm{D}(a, b, c)} = \frac{\mathrm{D}(X_0, Y_0, Z_0)}{\mathrm{D}(a, b, c)}, \tag{2.3}$$

where X_0, Y_0, Z_0 are the initial positions of the particles at $t = 0$. It is commonly assumed that the Lagrangian variables are identical to X_0, Y_0, Z_0; then the Jacobian in the right-hand side of (2.3) is unity. However, in this paper we shall hold to the most universal assumption that the Lagrangian variables are an arbitrary set of parameters. It allows us to perform any necessary change of variables a, b, c in the further analytical study. Accordingly, we keep the general form of (2.3).

It is well known that pressure and potential force can be eliminated from equations (2.2) without increasing the order of the system (see Stoker 1957; Lamb 1932). By cross-differentiation of equations (2.2) with respect to the Lagrangian variables we obtain a system that can be integrated over time to give

$$\left.\begin{aligned} X_{tb}X_c - X_{tc}X_b + Y_{tb}Y_c - Y_{tc}Y_b + Z_{tb}Z_c - Z_{tc}Z_b &= S_1(a, b, c), \\ X_{tc}X_a - X_{ta}X_c + Y_{tc}Y_a - Y_{ta}Y_c + Z_{tc}Z_a - Z_{ta}Z_c &= S_2(a, b, c), \\ X_{ta}X_b - X_{tb}X_a + Y_{ta}Y_b - Y_{tb}Y_a + Z_{ta}Z_b - Z_{tb}Z_a &= S_3(a, b, c). \end{aligned}\right\} \tag{2.4}$$

The integrals of motion S_1, S_2, S_3 found on the right-hand side of these equations are functions of Lagrangian variables. They are time-independent and are determined from the initial conditions. Similar equations were introduced in Cauchy's papers reviewed by Lamb (1932), and, therefore, S_1, S_2, S_3 are usually called the Cauchy invariants. Physically, conservation of these quantities follows from Kelvin's circulation theorem. The Cauchy invariants represent circulations around infinitesimal material circuits bounding surface elements which correspond to pairs of differentials $\mathrm{d}b\,\mathrm{d}c, \mathrm{d}a\,\mathrm{d}c, \mathrm{d}b\,\mathrm{d}a$ (we shall demonstrate this later in § 2.4). It follows immediately from (2.4) that the derivatives of Cauchy invariants S_1, S_2, S_3 with respect to the Lagrangian variables satisfy the equation

$$\frac{\partial S_1}{\partial a} + \frac{\partial S_2}{\partial b} + \frac{\partial S_3}{\partial c} = 0. \tag{2.5}$$

For irrotational motion all the Cauchy invariants vanish and so do the left-hand sides of (2.4).

2.2. *Jacobi matrix and matrix form of the governing equations*

In the Lagrangian representation governing equations are always formulated in terms of particle trajectories $\boldsymbol{X}(\boldsymbol{a}, t)$, where $\boldsymbol{a} = \{a, b, c\}$. Let us consider fluid motion from another viewpoint by focusing on relative displacements of neighbouring particles. We now study a material element $\mathrm{d}\boldsymbol{X} = \{\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z\}$ connecting two close particles and corresponding to the infinitesimal increment of Lagrangian variables $\mathrm{d}\boldsymbol{a} = \{\mathrm{d}a, \mathrm{d}b, \mathrm{d}c\}$. The Jacobi matrix is introduced to relate differentials $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{a}$ by

$$\mathrm{d}\boldsymbol{X} = \boldsymbol{R}\,\mathrm{d}\boldsymbol{a}, \quad \boldsymbol{R} = \begin{bmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{bmatrix} \tag{2.6}$$

and consists of the derivatives of current particle positions with respect to the Lagrangian variables. Like particle positions X, Y, Z, elements of the Jacobi matrix depend on the Lagrangian variables and time. We shall assume that the Lagrangian variables have the dimension of length and the elements R_{ij} are dimensionless. In the case of linear dependence of X, Y, Z on a, b, c, matrix $\boldsymbol{R}$ includes functions of time only. If initial particle positions X_0, Y_0, Z_0 are taken for the Lagrangian variables, then the time evolution of a material element $\mathrm{d}\boldsymbol{X}_0$ can be directly expressed as $\mathrm{d}\boldsymbol{X} = \boldsymbol{R}\,\mathrm{d}\boldsymbol{X}_0$, therefore matrix $\boldsymbol{R}$ is referred to as the distortion matrix by Landau *et al.* (1986). It is noteworthy that such a Jacobi matrix changes as a tensor with transformation of space coordinates X, Y, Z. Obviously, under the assumption $\boldsymbol{a} = \boldsymbol{X}_0$, the initial Jacobi matrix becomes a unit matrix.

Considering motion as continuous deformation, as is proposed in this paper, is natural for continuum mechanics (cf. the tensor description commonly employed in the elasticity theory). Let us determine how the Jacobi matrix is related to the strain tensor $\boldsymbol{E}$ defined in the elasticity theory (see Landau *et al.* 1986) by

$$|\mathrm{d}\boldsymbol{X}|^2 - |\mathrm{d}\boldsymbol{X}_0|^2 = 2\,E_{ij}\,\mathrm{d}a_i\,\mathrm{d}a_j.$$

In terms of the Jacobi matrix (2.6) the strain tensor is written as

$$\boldsymbol{E} = \tfrac{1}{2}\,(\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} - \boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0), \tag{2.7}$$

where $\boldsymbol{R}^{\mathrm{T}}$ is the transposed matrix. According to the above expression, the rotational part of motion is not accounted for when converting the Jacobi matrix into strain tensor. Hence, the latter cannot represent motion of a fluid adequately.

The present paper aims to show that the Jacobi matrix can play the role of a fundamental quantity sufficient to provide a closed description of fluid motion. Let us find out which equations it obeys. First note that the continuity equation (2.3) which requires constancy of the Jacobian in time naturally has a matrix form

$$\det\boldsymbol{R} = \det\boldsymbol{R}_0, \qquad \boldsymbol{R}_0 = \boldsymbol{R}\,|_{t=0}\,. \tag{2.8}$$

There is also the possibility to rewrite equations (2.4) in terms of the Jacobi matrix due to their homogeneous structure. It can be verified directly that (2.4) turns into a single matrix equation

$$\boldsymbol{R}_t^{\mathrm{T}}\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}_t = \boldsymbol{S}, \qquad \boldsymbol{S} = \begin{bmatrix} 0 & S_3 & -S_2 \\ -S_3 & 0 & S_1 \\ S_2 & -S_1 & 0 \end{bmatrix}. \tag{2.9}$$

The right-hand side of this matrix equation takes the form of an antisymmetric matrix $\boldsymbol{S}$ composed of the Cauchy invariants.

The pair of equations (2.8)–(2.9) represents a matrix analogue of (2.3)–(2.4) and is expressed in terms of the Jacobi matrix, that is of a set of nine unknown functions of the Lagrangian variables and time. Obviously, (2.8) and (2.9) are not sufficient for formulation of a closed system of governing matrix equations. Note that the matrix that satisfies (2.8) and (2.9) does not necessarily take the form of the Jacobi matrix (2.6), hence not every solution of these equations will be physically meaningful. To construct a closed set of hydrodynamic matrix equations we need to add a consistency condition to (2.8)–(2.9). For a consistent solution $\boldsymbol{R}$ we should be able to reconstruct trajectories $\boldsymbol{X}(\boldsymbol{a}, t)$, for which $\boldsymbol{R}$ serves as the Jacobi matrix.† Since, according to (2.6), rows of matrix $\boldsymbol{R}$ must contain components of gradients of X, Y, Z with respect to

† Supplementary conditions of the same nature are found in the elasticity theory where they are responsible for the presence or absence of dislocations (see Landau *et al.* 1986).

a, b, c, the consistency condition becomes

$$\frac{\partial R_{nm}}{\partial a_k} = \frac{\partial R_{nk}}{\partial a_m}, \tag{2.10}$$

all indices changing from 1 to 3 (we shall imply hereinafter when using indexed notation that a_1, a_2, a_3 and X_1, X_2, X_3 are identical to a, b, c and X, Y, Z, respectively). The local condition (2.10) is equivalent to vanishing of integral $\oint \boldsymbol{R}\, \mathrm{d}\boldsymbol{a}$ around any closed curve in the space of Lagrangian variables, hence, integrals $\int_A^B \boldsymbol{R}\, \mathrm{d}\boldsymbol{a}$, where A and B are arbitrary points in space a, b, c, must not depend on the path between A and B.

For solution $\boldsymbol{R}$ of the complete system (2.8)–(2.10) particle trajectories are obtained by integration over the Lagrangian variables

$$\boldsymbol{X}(\boldsymbol{a}, t) = \int_0^{\boldsymbol{a}} \boldsymbol{R}(\boldsymbol{q}, t)\mathrm{d}\boldsymbol{q} + \boldsymbol{X}(0, t). \tag{2.11}$$

The integration constant $\boldsymbol{X}(0, t)$ is an arbitrary function of time independent of the Lagrangian variables. This term is related only to the motion of fluid mass as a whole and does not change the relative motion of individual particles. The effect of this term is the same as of observing the flow from a certain, generally non-inertial, frame of reference. As seen from the basic equations (2.2) and (2.3), this 'solid-body' component does not influence non-trivial motion in the mass of fluid and therefore it will be omitted.

Thus, the matrix equations of motion (2.9) and continuity (2.8) together with the consistency condition (2.10) constitute a closed formulation of hydrodynamic equations that is equivalent to the initial equations (2.2), (2.3) and must be solved for the Jacobi matrix.

Several general features of (2.8)–(2.10) are of importance. Equation (2.9) contains time derivatives of $\boldsymbol{R}$ only and is free of a, b, c-derivatives. On the other hand, the consistency condition (2.10) is a linear equation for space derivatives where time does not appear at all.

As in the other approaches, a particular problem may be greatly simplified in the matrix formulation by an appropriate choice of the coordinate system and the Lagrangian variables. We now discuss how a change of variables can modify the form of matrix equations. It is noteworthy that linear unitary transformations of X, Y, Z and a, b, c to complex coordinates and Lagrangian variables do not change the structure of matrix equations. For instance, let us introduce complex coordinates $\boldsymbol{W} = \{W_1, W_2, W_3\}$ defined by

$$W_1 = 2^{-1/2}(X + \mathrm{i}Y), \; W_2 = 2^{-1/2}(X - \mathrm{i}Y), \; W_3 = Z \tag{2.12}$$

in place of the real quantities $\boldsymbol{X} = \{X, Y, Z\}$. The vector form of this transformation involves the unitary matrix $\boldsymbol{T}$:

$$\boldsymbol{W} = \boldsymbol{T}\boldsymbol{X}, \quad \text{where} \quad \boldsymbol{T} = 2^{-1/2} \begin{bmatrix} 1 & \mathrm{i} & 0 \\ 1 & -\mathrm{i} & 0 \\ 0 & 0 & 2^{1/2} \end{bmatrix}. \tag{2.13}$$

This particular form of $\boldsymbol{T}$ does not restrict substantially the generality of further analysis. As is known, an arbitrary unitary transformation can be decomposed into the rotation of axes by a certain angle and the unitary transformation (2.13). The derivatives of the new coordinates with respect to a, b, c can be considered as elements of a new complex Jacobi matrix $\boldsymbol{L}_1 = (\partial W_i/\partial a_j) = \boldsymbol{TR}$. The system of hydrodynamic equations in terms of $\boldsymbol{L}_1$ takes the form

$$\det \boldsymbol{L}_1 = \det \boldsymbol{L}_{1\,0}, \tag{2.14a}$$

$$\boldsymbol{L}^*_{1\,t}\boldsymbol{L}_1 - \boldsymbol{L}^*_1\boldsymbol{L}_{1\,t} = \boldsymbol{S}, \tag{2.14b}$$

$$\frac{\partial L_{1\,nm}}{\partial a_k} = \frac{\partial L_{1\,nk}}{\partial a_m}, \tag{2.14c}$$

where $\boldsymbol{L}^*_1$ denotes the Hermitian conjugate to $\boldsymbol{L}_1$. The right-hand side of (2.14*a*) is $\det \boldsymbol{L}_{1\,0} = -\mathrm{i} \det \boldsymbol{R}_0$, the matrix $\boldsymbol{S}$ in (2.14*b*) must be real and antisymmetric as in the original equation (2.9). However, only those complex solutions $\boldsymbol{L}_1$ of the system (2.14*a*–*c*) that can be conversely transformed into a real $\boldsymbol{R}$ satisfying equations (2.8)–(2.10) are physically significant. To separate unnecessary complex solutions, an additional restriction on $\boldsymbol{L}_1$ should be imposed. Namely, it requires that reconversion of $\boldsymbol{L}_1$ to the original real variables should result in a real-valued matrix: $\mathrm{Im}\{\boldsymbol{T}^*\boldsymbol{L}_1\} = 0$.

Another way of modifying system (2.8)–(2.10) is an appropriate choice of the Lagrangian variables. It is even possible to employ complex parameters as the Lagrangian variables, as it was shown by Abrashkin & Yakubovich (1984) for two-dimensional flows. Consider, for instance, the form of matrix equations (2.8)–(2.10) after conversion to the complex Lagrangian variables $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ defined by

$$\xi_1 = 2^{-1/2}(a + \mathrm{i}b),\ \xi_2 = 2^{-1/2}(a - \mathrm{i}b),\ \xi_3 = c \quad \text{or} \quad \boldsymbol{\xi} = \boldsymbol{T}\boldsymbol{a}, \tag{2.15}$$

where $\boldsymbol{T}$ is the same as in (2.13). With this change of variables, a new Jacobi matrix can be defined as $\boldsymbol{L}_2 = (\partial X_i/\partial \xi_j) = \boldsymbol{RT}^*$. Matrix $\boldsymbol{L}_2$ obeys a system of hydrodynamic equations almost identical to (2.14*a*–*c*). The difference is that in place of $\boldsymbol{S}$ in (2.14*b*), the complex anti-Hermitian matrix $\boldsymbol{TST}^*$ appears on the right-hand side of the new equation of motion, and the consistency condition should be rewritten with the new complex Lagrangian variables: $\partial L_{2\,nm}/\partial \xi_k = \partial L_{2\,nk}/\partial \xi_m$.

Now let us apply both transformations to the complex space coordinates (2.13) and complex Lagrangian variables (2.15) together†. In this case the complex Jacobi matrix is introduced as $\boldsymbol{L} = (\partial W_i/\partial \xi_j) = \boldsymbol{TRT}^*$ and obeys the system of hydrodynamic equations

$$\det \boldsymbol{L} = \det \boldsymbol{L}_0, \tag{2.16a}$$

$$\boldsymbol{L}^*_t\boldsymbol{L} - \boldsymbol{L}^*\boldsymbol{L}_t = \boldsymbol{M}, \tag{2.16b}$$

$$\frac{\partial L_{nm}}{\partial \xi_k} = \frac{\partial L_{nk}}{\partial \xi_m}, \tag{2.16c}$$

where $\boldsymbol{M} = \boldsymbol{TST}^*$ is a complex anti-Hermitian matrix independent of time. The matrix $\boldsymbol{M}$ can be expressed in terms of the Cauchy invariants S_1, S_2, S_3 (see (2.4), (2.9)) as follows:

$$\boldsymbol{M} = \begin{bmatrix} -\mathrm{i}S_3 & 0 & 2^{-1/2}(-S_2 + \mathrm{i}S_1) \\ 0 & \mathrm{i}S_3 & 2^{-1/2}(-S_2 - \mathrm{i}S_1) \\ 2^{-1/2}(S_2 + \mathrm{i}S_1) & 2^{-1/2}(S_2 - \mathrm{i}S_1) & 0 \end{bmatrix}. \tag{2.17}$$

† In general, the transformation matrices for coordinates and Lagrangian variables may differ, but it would not change the structure of the resulting system of matrix equations. Here, we consider transformations $\boldsymbol{W} = \boldsymbol{T}\boldsymbol{X}$ and $\boldsymbol{\xi} = \boldsymbol{T}\boldsymbol{a}$ with the same $\boldsymbol{T}$ for simplicity of manipulation only.

The restriction on a solution to (2.16*a–c*) similar to that established for (2.14*a–c*) remains valid. Namely, the reconversion of a solution $\boldsymbol{L}$ to the real variables must yield a real matrix: $\mathrm{Im}\{\boldsymbol{T}^*\boldsymbol{L}\boldsymbol{T}\} = 0$. To obtain explicit conditions following from this requirement, we write down elements of $\boldsymbol{L}$ in terms of X, Y, Z and a, b, c using (2.12), (2.15):

$$\boldsymbol{L} = \begin{bmatrix} \dfrac{\partial(X+\mathrm{i}Y)}{\partial(a+\mathrm{i}b)} & \dfrac{\partial(X+\mathrm{i}Y)}{\partial(a-\mathrm{i}b)} & \dfrac{\partial(X+\mathrm{i}Y)}{\partial c} \\ \dfrac{\partial(X-\mathrm{i}Y)}{\partial(a+\mathrm{i}b)} & \dfrac{\partial(X-\mathrm{i}Y)}{\partial(a-\mathrm{i}b)} & \dfrac{\partial(X-\mathrm{i}Y)}{\partial c} \\ \dfrac{\partial Z}{\partial(a+\mathrm{i}b)} & \dfrac{\partial Z}{\partial(a-\mathrm{i}b)} & \dfrac{\partial Z}{\partial c} \end{bmatrix}. \tag{2.18}$$

Obviously, the elements of $\boldsymbol{L}$ must be interrelated as follows:

$$L_{22} = \bar{L}_{11}, \qquad L_{12} = \bar{L}_{21}, \qquad L_{23} = \bar{L}_{13}, \qquad L_{32} = \bar{L}_{31}, \qquad \mathrm{Im}\, L_{33} = 0, \tag{2.19}$$

where the bar denotes a complex conjugate. For the chosen set of complex coordinates and Lagrangian variables, any matrix $\boldsymbol{L}$ satisfying the conditions (2.19) corresponds to a real matrix $\boldsymbol{R} = \boldsymbol{T}^*\boldsymbol{L}\boldsymbol{T}$.

The system of equations for the Jacobi matrix should be solved under certain initial and boundary conditions.

2.3. *Initial and boundary conditions*

We now specify the formulation of initial-value problems of fluid dynamics for real matrix equations. Although the initial matrix $\boldsymbol{R}_0$ also appears on the right-hand side of the continuity equation, it is virtually only related to the choice of the Lagrangian variables a, b, c for a particular problem and does not contain any information about the initial flow pattern. Obviously, $\boldsymbol{R}_0$ is supposed to satisfy the condition (2.10). Physically meaningful initial conditions consist in a given velocity field at $t = 0$, and therefore the initial time derivative $\boldsymbol{R}_{t0}$ that specifies velocities at $t = 0$ is essential. A particular feature of the matrix equations (2.8), (2.9) is that the time invariants determined by the initial matrices $\boldsymbol{R}_0$ and $\boldsymbol{R}_{t0}$ appear on their right-hand sides. For the matrix of the Cauchy invariants $\boldsymbol{S}$, by virtue of (2.9) we have

$$\boldsymbol{S} = \boldsymbol{R}_{t0}^{\mathrm{T}}\boldsymbol{R}_0 - \boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_{t0}.$$

We consider now the boundary conditions for the matrix equations. Generally, it is impossible to give their closed expression in terms of the Jacobi matrix.† We shall therefore revert to the trajectories $\boldsymbol{X}(\boldsymbol{a}, t)$ related to $\boldsymbol{R}$ by (2.11) and survey the Lagrangian form of the ideal-fluid boundary conditions. This form was mentioned in Stoker (1957) but still requires a more systematic discussion. There are three primary boundary problems of fluid dynamics: a flow at a rigid boundary, free-surface problems, and flows with interfaces between neighbouring motions of different types.

We begin with the impermeability condition at the boundary of a solid body. In the most general case, the surface of the body bounding the flow can be described by the implicit equation $Q(\boldsymbol{X}) = 0$, where Q is a scalar function of coordinate $\boldsymbol{X}$. As is well known, the impermeable boundary conditions at this surface are formulated for an ideal fluid as $\boldsymbol{X}_t \cdot \nabla Q = 0$. Considering $Q(\boldsymbol{X}(\boldsymbol{a}, t))$ as a function of $\boldsymbol{a}$ we find

† It is quite analogous, say, to Helmholtz's equation of vorticity (see e.g. Saffman 1992, § 1.5), where there are no explicit inviscid boundary conditions for vorticity, and velocity field should be determined to impose the conventional boundary conditions.

that $\boldsymbol{X}_t \cdot \nabla Q = (\partial Q/\partial X_i)(\partial X_i/\partial t) = (\partial Q/\partial t)|_{\boldsymbol{a}=\text{const}} = 0$, that is, Q as a function of a, b, c must be conserved for any particle travelling along the surface of the body. Thus, if the rigid boundary has no cusps and sharp edges and flow separation does not occur, so that fluid particles do not come onto the boundary and do not leave it, then $(\partial Q/\partial t)|_{\boldsymbol{a}=\text{const}} = 0$ can serve as the Lagrangian condition of impermeability.

At the free surface of a fluid, both kinematic (impermeability of the surface) and dynamic (constancy of pressure) conditions must be satisfied, the shape of the surface being not fixed contrary to the previous case. In the Lagrangian approach, the kinematic condition is reduced to the statement that the free surface corresponds to a fixed surface $Q(\boldsymbol{a})$ in the space of the Lagrangian variables, which is simpler and more straightforward than the conventional Eulerian formulation. The commonly used dynamic condition of pressure constancy at the free surface $p|_{Q=0} = \text{const}$ can be converted into the condition of collinearity of the gradients of pressure and of function $Q(\boldsymbol{a})$ with respect to the Lagrangian variables

$$\nabla_{\boldsymbol{a}}\, p \times \nabla_{\boldsymbol{a}}\, Q = 0,$$

where $\nabla_{\boldsymbol{a}}\, p = -\rho\ (\boldsymbol{R}^{\mathrm{T}}\boldsymbol{X}_{tt} + \nabla_{\boldsymbol{a}}\, \Phi)$ from the initial equation of motion (2.2), and acceleration $\boldsymbol{X}_{tt}$ can be found from (2.11) by time differentiation.

Consider now the third type of boundary condition, that across the interfaces separating motions of different types, for instance potential and rotational flows or domains of closed trajectories adjacent to regions of unbounded pathlines going to infinity. The Lagrangian variables $\boldsymbol{q}_{1,2}$ in these regions can be introduced in different ways through functions $\boldsymbol{X}_{10}(\boldsymbol{q}_1)$ and $\boldsymbol{X}_{20}(\boldsymbol{q}_2)$, indices 1 and 2 denoting the domain of the flow. But the equations of the interface must of course coincide while coordinates $\boldsymbol{X}_1, \boldsymbol{X}_2$ tend to the interface: $Q(\boldsymbol{X}_1, t) = Q(\boldsymbol{X}_2, t) = 0$, even if $\boldsymbol{X}_1(\boldsymbol{q}_1, t)$ and $\boldsymbol{X}_2(\boldsymbol{q}_2, t)$ are different functions of the Lagrangian variables at the interface. When the normal components of velocity $\boldsymbol{V} = \boldsymbol{X}_t$ are regarded as functions of space coordinates, they must be continuous across the interface:

$$V_{1n} = V_{2n}, \qquad V_{1,2\,n} = \boldsymbol{X}_{1,2\,t} \cdot \nabla_X\, Q\ |\nabla_X\, Q|^{-1}, \tag{2.20a,b}$$

where ∇_X is the gradient with respect to coordinates X, Y, Z. If a jump in the tangential velocity is admitted, the continuity condition at the interface should be satisfied for both normal velocity and pressure. Under the assumption of no jump in the tangential velocity, the condition (2.20*a*) for the normal velocity and, additionally, a similar condition for the tangential velocity must hold. It is practical to introduce the Lagrangian variables in two adjacent regions consistently in order to obtain parameterizations that coincide: $\boldsymbol{q}_1 = \boldsymbol{q}_2$, $\boldsymbol{X}_1(\boldsymbol{q}_1, t) = \boldsymbol{X}_2(\boldsymbol{q}_2, t)$ while coordinates $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ tend to the interface. With such parameterizations, velocity components in (2.20*a*) can be expressed as functions of the Lagrangian variables. A similar technique will be employed in § 4 when considering potential and rotational motions that agree at the boundary of a vortex.

2.4. *Evolution of vorticity in matrix representation*

The vorticity field is an essential characteristic of any motion of a fluid. We shall now discuss how vorticity dynamics is expressed in terms of the Jacobi matrix. Together with vector $\boldsymbol{\Omega} = \operatorname{rot} \boldsymbol{V}$, consider vorticity matrix $\mathbf{\Omega}$ defined via the antisymmetric part of the matrix of velocity derivatives with respect to coordinates $(\partial V_i/\partial X_j)$, so that the elements of matrix $\mathbf{\Omega}$ coincide with vorticity components:

$$(\mathbf{\Omega})_{ij} = \frac{\partial V_j}{\partial X_i} - \frac{\partial V_i}{\partial X_j}. \tag{2.21}$$

An expression for vorticity components Ω_i and elements of the vorticity matrix $(\mathbf{\Omega})_{ij}$ is obtained using the Levi–Civita tensor e_{ijk}:

$$(\mathbf{\Omega})_{ij} = e_{ijk}\,\Omega_k, \quad \Omega_i = \tfrac{1}{2}\,e_{ijk}\,(\mathbf{\Omega})_{jk}. \tag{2.22a,b}$$

On substituting the equality $(\partial V_i/\partial X_j) = \boldsymbol{R}_t\boldsymbol{R}^{-1}$ into (2.21) the expression for matrix $\mathbf{\Omega}$ in terms of the Jacobi matrix becomes

$$\mathbf{\Omega} = \left(\boldsymbol{R}^{-1}\right)^{\mathrm{T}}\boldsymbol{R}_t^{\mathrm{T}} - \boldsymbol{R}_t\boldsymbol{R}^{-1}. \tag{2.23}$$

The relationship between $\mathbf{\Omega}$ and the matrix of the Cauchy invariants, $\boldsymbol{S}$, is found by premultiplying (2.23) by $\boldsymbol{R}^{\mathrm{T}}$ and postmultiplying it by $\boldsymbol{R}$, and then by using the matrix equation of motion (2.9):

$$\boldsymbol{R}^{T}\mathbf{\Omega}\,\boldsymbol{R} = \boldsymbol{S}, \quad \mathbf{\Omega} = \left(\boldsymbol{R}^{-1}\right)^{\mathrm{T}}\boldsymbol{S}\,\boldsymbol{R}^{-1}. \tag{2.24a,b}$$

We see from these matrix equations that the vorticity can be obtained directly from the Jacobi matrix, provided the Cauchy invariants are known. Let us introduce a vector notation for the set of Cauchy invariants: $\boldsymbol{S} = \{S_1, S_2, S_3\}$, so that the relationship between $\boldsymbol{S}$† and matrix $\boldsymbol{S}$ takes the same form as the relationship (2.22b) between vorticity and matrix $\mathbf{\Omega}$. Reduction of the matrix equation (2.24b) to the vector form is performed using the Levi–Civita tensor and results in

$$\boldsymbol{\Omega} = \frac{\boldsymbol{R}}{\det\boldsymbol{R}_0}\boldsymbol{S}, \tag{2.25}$$

which is, in fact, the well-known Cauchy equation for vorticity (Batchelor 1967; Saffman 1992, § 1.7) in a general form for arbitrary choice of the Lagrangian variables. The conventional form of this equation

$$\boldsymbol{\Omega} = \boldsymbol{R}\,\boldsymbol{\Omega}_0, \quad \text{where} \quad \boldsymbol{R} = \left(\frac{\partial X_i}{\partial X_{0j}}\right), \tag{2.26}$$

implies that the initial particle positions X_0, Y_0, Z_0 are taken as the Lagrangian variables.

The generalized equation (2.25) enables one to find the Cauchy invariants for arbitrary a, b, c from given initial vorticity

$$\boldsymbol{S} = \boldsymbol{R}_0^{-1}\,\boldsymbol{\Omega}_0\,\det\boldsymbol{R}_0.$$

Obviously, in the particular case $\boldsymbol{a} = \boldsymbol{X}_0$ ($\boldsymbol{R}_0 = \boldsymbol{I}$, $\det\boldsymbol{R}_0 = 1$), the Cauchy invariants are identical to the initial vorticity components: $\boldsymbol{S} = \boldsymbol{\Omega}_0$.

The expanded form of (2.25)

$$\boldsymbol{\Omega} = \frac{1}{\det\boldsymbol{R}_0}\left(S_1\frac{\partial\boldsymbol{X}}{\partial a} + S_2\frac{\partial\boldsymbol{X}}{\partial b} + S_3\frac{\partial\boldsymbol{X}}{\partial c}\right) \tag{2.27}$$

allows us to interpret the meaning of the Cauchy invariants geometrically. Consider a, b, c as the coordinates of a curvilinear coordinate system in space. By definition of the Lagrangian variables, such a 'frozen-in' coordinate system must experience

† To avoid misunderstanding, we note that $\boldsymbol{S}$ is not a true vector in the physical space X, Y, Z. The physical meaning of S_1, S_2, S_3 will be discussed later.

continuous deformation in the course of fluid motion. It should be noted that vectors $\partial \boldsymbol{X}/\partial a$, $\partial \boldsymbol{X}/\partial b$, $\partial \boldsymbol{X}/\partial c$ indicate the directions of displacement of the point $\boldsymbol{X}(\boldsymbol{a}, t)$ during the variation of a, b, c, that is, the directions of the corresponding coordinate lines in space. Therefore, we can see from (2.27) that $S_i / \det \mathbf{R}_0$ are, actually, components of vorticity along the frozen-in 'axes' of Lagrangian variables. Obviously, the total vorticity field also proves to be frozen-in due to constancy of these components in time.

An important particular case, when only one of the three Cauchy invariants, say, S_3, is not zero so that $\boldsymbol{S} = \{0, 0, S_3\}$, is worthy of more detailed consideration. Under this assumption, expression (2.27) reduces to

$$\boldsymbol{\Omega} = \frac{S_3}{\det \mathbf{R}_0} \frac{\partial \boldsymbol{X}}{\partial c}. \tag{2.28}$$

It follows from vanishing of divergence of $\boldsymbol{S}$ with respect to the Lagrangian variables (2.5) that the only non-zero Cauchy invariant does not depend on c: $S_3 = S_3(a, b)$. Besides, the vorticity (2.28) is everywhere tangential to the coordinate lines of c: $\boldsymbol{\Omega} \parallel \boldsymbol{X}_c$. This means that for such flows the vortex lines containing a given particle $\boldsymbol{a}_* = \{a_*, b_*, c_*\}$ can be immediately found from particle trajectories $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{a}, t)$ by varying c, with a_* and b_* fixed.

Another noteworthy feature of flows with one non-zero Cauchy invariant only can be also seen from (2.28). It turns out that the Lagrangian variables for such flows are closely connected with Clebsch potentials (see Lamb 1932), which serve as the canonical variables in the Hamiltonian formulation of fluid dynamics (Zakharov & Kuznetsov 1997). To demonstrate this we employ the vector identity $\boldsymbol{X}_c = \det \mathbf{R}_0 \nabla a \times \nabla b$ following from the fact that the Jacobi matrix $(\partial X_i/\partial a_j)$ is inverse to the matrix $(\partial a_i/\partial X_j)$. On substituting this identity into (2.28), for vorticity we have

$$\boldsymbol{\Omega} = S_3 \nabla a \times \nabla b = \nabla \lambda \times \nabla \mu,$$

where λ and μ stand for the Clebsch potentials which can be defined as $\lambda = a$, $\mu = \int S_3 \, \mathrm{d}b$ or as $\lambda = \int S_3 \, \mathrm{d}a$, $\mu = b$.

An alternative physical interpretation of the Cauchy invariants is provided by resolving (2.25) for $\boldsymbol{S}$ and then expanding the result into component form:

$$S_1 = \det \mathbf{R}_0 \nabla a \cdot \boldsymbol{\Omega}, \quad S_2 = \det \mathbf{R}_0 \nabla b \cdot \boldsymbol{\Omega}, \quad S_3 = \det \mathbf{R}_0 \nabla c \cdot \boldsymbol{\Omega}. \tag{2.29a–c}$$

Since elements $\partial a_i/\partial X_j$ constitute a matrix which is inverse to the Jacobi matrix $(\partial X_i/\partial a_j)$, it is possible to derive the vector formulae $\det \mathbf{R}_0 \nabla a = \boldsymbol{X}_b \times \boldsymbol{X}_c$, $\det \mathbf{R}_0 \nabla b = \boldsymbol{X}_c \times \boldsymbol{X}_a$, $\det \mathbf{R}_0 \nabla c = \boldsymbol{X}_a \times \boldsymbol{X}_b$, which can then be substituted into (2.29*a–c*). Now consider the infinitesimal surface elements corresponding to differentials of the Lagrangian variables $\mathrm{d}b\,\mathrm{d}c$, $\mathrm{d}c\,\mathrm{d}a$ and $\mathrm{d}a\,\mathrm{d}b$, which are characterized by normal vectors $\boldsymbol{X}_b \times \boldsymbol{X}_c \, \mathrm{d}b\,\mathrm{d}c$, $\boldsymbol{X}_c \times \boldsymbol{X}_a \, \mathrm{d}c\,\mathrm{d}a$, $\boldsymbol{X}_a \times \boldsymbol{X}_b \, \mathrm{d}a\,\mathrm{d}b$ of magnitudes equal to the areas of the surface elements. It follows from (2.29*a–c*) and the vector identities mentioned above that the Cauchy invariants represent vorticity fluxes through these surface elements or, by the Stokes theorem, elementary circulations $\mathrm{d}\Gamma_i$ around material circuits embracing the surface elements:

$$\mathrm{d}\Gamma_1 = S_1 \, \mathrm{d}b\,\mathrm{d}c, \quad \mathrm{d}\Gamma_2 = S_2 \, \mathrm{d}c\,\mathrm{d}a, \quad \mathrm{d}\Gamma_3 = S_3 \, \mathrm{d}a\,\mathrm{d}b.$$

Thus, constancy of S_i reflects Kelvin's circulation theorem locally.

3. Exact solutions to matrix equations: flows with precessing vorticity

3.1. *Elementary solutions to matrix equations*

As an example of an elementary solution to the hydrodynamic equations (2.8)–(2.10) let us consider the Jacobi matrix

$$\boldsymbol{R} = \mathrm{e}^{\hat{\gamma}t}\,\boldsymbol{R}_0, \tag{3.1}$$

where $\hat{\gamma}$ is a constant (independent of a, b, c, t) antisymmetric matrix

$$\hat{\gamma} = \begin{bmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{bmatrix}. \tag{3.2}$$

On substituting this matrix expression into (2.8)–(2.10), we first note that the determinant of (3.1) does not depend on time by virtue of orthogonality of the matrix exponential $\mathrm{e}^{\hat{\gamma}t}$: $\det \mathrm{e}^{\hat{\gamma}t} = 1$, $\det \boldsymbol{R} = \det \boldsymbol{R}_0$, so the continuity equation (2.8) is satisfied identically. It is natural to assume that the initial matrix $\boldsymbol{R}_0$ defined by the choice of the set of Lagrangian variables obeys the consistency conditions (2.10). By direct verification we see that under this supposition for $\boldsymbol{R}_0$ the conditions (2.10) hold for any subsequent moment of time. Finally, the equation of motion (2.9) is satisfied, with the matrix of the Cauchy invariants being

$$\boldsymbol{S} = -2\,\boldsymbol{R}_0^{\mathrm{T}}\hat{\gamma}\,\boldsymbol{R}_0. \tag{3.3}$$

So, solution (3.1) describes the evolution of any (not necessarily infinitesimal) material element $\delta\boldsymbol{X}_0$ as $\delta\boldsymbol{X} = \exp(\hat{\gamma}t)\,\delta\boldsymbol{X}_0$. It can be seen that (3.1) implies the solid-body rotation of the fluid as a whole. Indeed, $(\delta\boldsymbol{X})_t = \hat{\gamma}\,\delta\boldsymbol{X} = -\boldsymbol{\gamma}\times\delta\boldsymbol{X}$ that corresponds to rotation with angular velocity $-\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$. The vorticity of such a motion is constant and uniform: $\boldsymbol{\Omega} = -2\boldsymbol{\gamma}$. Note that the Cauchy invariants in (3.3) depend on the form of the initial matrix $\boldsymbol{R}_0$, that is, on the choice of the Lagrangian variables.

The system of matrix equations also admits solutions

$$\boldsymbol{R} = \boldsymbol{R}_0\,\mathrm{e}^{\hat{\omega}t}, \quad (\hat{\omega})_{ij} = e_{ijk}\omega_k, \;\; \omega_k = \text{const}, \tag{3.4}$$

where matrix $\hat{\omega}$ is constant and antisymmetric, as in (3.1).

Unlike solution (3.1), this matrix expression satisfies (2.9) only if the matrices $\hat{\omega}$ and $\boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0$ commute. The matrix of the Cauchy invariants for such solutions takes the form

$$\boldsymbol{S} = -2\,\hat{\omega}\,\boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0 \tag{3.5}$$

and also commutes with $\hat{\omega}$. The consistency conditions (2.10) impose additional constraints on $\boldsymbol{R}_0$. We shall analyse this type of solution later as a particular case of a more general family.

Let us now seek solutions to the governing equations (2.8)–(2.10) by combining expressions (3.1) and (3.4) into

$$\boldsymbol{R} = \mathrm{e}^{\hat{\gamma}t}\,\boldsymbol{R}_0\,\mathrm{e}^{\hat{\omega}t}, \tag{3.6}$$

where the antisymmetrical matrices $\hat{\gamma}$ and $\hat{\omega}$ consist of constants, as in (3.2) and (3.4). This expression contains three matrix co-factors, two exponential terms depending on time only, and $\boldsymbol{R}_0$ determined by the relationship between X_0, Y_0, Z_0 and a, b, c. It is obvious that the continuity equation (2.8) is satisfied by (3.6) identically.

Let us analyse the equation of motion (2.9) for the matrix expression (3.6). On substitution we find that, if the commutation condition

$$\left[\hat{\boldsymbol{\omega}},\ \left(2\boldsymbol{R}_0^{\mathrm{T}}\hat{\boldsymbol{\gamma}}\,\boldsymbol{R}_0 + \hat{\boldsymbol{\omega}}\,\boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0 + \boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0\,\hat{\boldsymbol{\omega}}\right)\right] = 0 \tag{3.7}$$

holds, then matrix $\boldsymbol{S}$ does not depend on time and (2.9) is satisfied. Then, for matrix $\boldsymbol{S}$ we have

$$\boldsymbol{S} = -2\boldsymbol{R}_0^{\mathrm{T}}\hat{\boldsymbol{\gamma}}\,\boldsymbol{R}_0 - \hat{\boldsymbol{\omega}}\,\boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0 - \boldsymbol{R}_0^{\mathrm{T}}\boldsymbol{R}_0\,\hat{\boldsymbol{\omega}}. \tag{3.8}$$

At the same time, the commutation condition (3.7) imposes a certain constraint on the form of the initial matrix $\boldsymbol{R}_0$. Its analysis is greatly simplified by converting matrices $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\omega}}$ to the diagonal form. Transformation of matrix $\hat{\boldsymbol{\gamma}}$ into the form $\mathrm{diag}\{-\mathrm{i}\gamma, \mathrm{i}\gamma, 0\}$, γ being the modulus of its non-zero eigenvalues, is performed by rotation of coordinate axes X, Y, Z with subsequent change to complex variables by formulae (2.12). A similar transformation of the Lagrangian variables, with the change to the complex variables introduced in (2.15), allows us to independently diagonalize matrix $\hat{\boldsymbol{\omega}}$ to $\mathrm{diag}\{-\mathrm{i}\omega, \mathrm{i}\omega, 0\}$, where ω stands for the absolute value of the non-zero eigenvalues of the initial real matrix $\hat{\boldsymbol{\omega}}$. As we have seen in § 2.2, the combined transformations of X, Y, Z and the Lagrangian variables imply conversion of real matrix $\boldsymbol{R}$ into complex Jacobi matrix $\boldsymbol{L}$, that must satisfy system (2.16*a–c*). A solution sought in the form (3.6) corresponds to the complex matrix

$$\boldsymbol{L} = \mathrm{diag}\{\mathrm{e}^{-\mathrm{i}\gamma t},\ \mathrm{e}^{\mathrm{i}\gamma t},\ 1\}\,\boldsymbol{L}_0\,\mathrm{diag}\{\mathrm{e}^{-\mathrm{i}\omega t},\ \mathrm{e}^{\mathrm{i}\omega t},\ 1\}. \tag{3.9}$$

On substituting it into the complex motion equation (2.16*b*) we obtain on the left-hand side the expression $\mathrm{e}^{-\hat{\boldsymbol{\omega}}t}\boldsymbol{N}\,\mathrm{e}^{\hat{\boldsymbol{\omega}}t}$, where matrix $\boldsymbol{N}$ is given by

$$\boldsymbol{N} = -2\ \boldsymbol{L}_0^{*}\hat{\boldsymbol{\gamma}}\,\boldsymbol{L}_0 - \hat{\boldsymbol{\omega}}\,\boldsymbol{L}_0^{*}\boldsymbol{L}_0 - \boldsymbol{L}_0^{*}\boldsymbol{L}_0\,\hat{\boldsymbol{\omega}}. \tag{3.10}$$

Further development shows that $\boldsymbol{N}$ contains only two independent elements N_{11}, N_{31} and has the following structure:

$$\boldsymbol{N} = \begin{bmatrix} N_{11} & 0 & -\bar{N}_{31} \\ 0 & -N_{11} & -N_{31} \\ N_{31} & \bar{N}_{31} & 0 \end{bmatrix}, \tag{3.11a}$$

where

$$N_{11} = 2\mathrm{i}\left\{(\omega+\gamma)|L_{11}|^2 + (\omega-\gamma)|L_{21}|^2 + \omega|L_{31}|^2\right\}, \tag{3.11b}$$

$$N_{31} = \mathrm{i}\left\{(\omega+2\gamma)L_{11}\bar{L}_{13} + (\omega-2\gamma)L_{21}L_{13} + \omega L_{31}L_{33}\right\}; \tag{3.11c}$$

L_{nm} denote the elements of $\boldsymbol{L}_0$ (the subscript '0' in $L_{0,nm}$ is omitted for brevity). For the matrix of the Cauchy invariants $\boldsymbol{M} = \mathrm{e}^{-\hat{\boldsymbol{\omega}}t}\boldsymbol{N}\,\mathrm{e}^{\hat{\boldsymbol{\omega}}t}$ to be independent of time, it is necessary and sufficient that matrices $\boldsymbol{N}$ and $\hat{\boldsymbol{\omega}} = \mathrm{diag}\{-\mathrm{i}\omega, \mathrm{i}\omega, 0\}$ should commute. Therefore, $\boldsymbol{N}$ must have the diagonal structure as well as $\hat{\boldsymbol{\omega}}$. Vanishing of N_{31} ensures that all non-diagonal elements of $\boldsymbol{N}$ are equal to zero. So, $N_{31} = 0$ will be the requirement on the initial Jacobi matrix $\boldsymbol{L}_0$ sought that follows from the equation of motion.

In addition, to construct a solution of the complete set of matrix equations, we must satisfy the consistency condition (2.16*c*). Suppose that matrix $\boldsymbol{L}_0$ in (3.9) has the following structure: elements of any kth column depend only on the kth Lagrangian variable:

$$\boldsymbol{L}_0 = \begin{bmatrix} L_{11}(\xi_1) & L_{12}(\xi_2) & L_{13}(c) \\ L_{21}(\xi_1) & L_{22}(\xi_2) & L_{23}(c) \\ L_{31}(\xi_1) & L_{32}(\xi_2) & L_{33}(c) \end{bmatrix}, \tag{3.12}$$

where all elements L_{nm} are functions of one argument, and c is identical to the real variable ξ_3 according to (2.15). If such a separation of variables in columns occurs, then the consistency condition for $\boldsymbol{L}_0$ (2.16*c*) is satisfied trivially:

$$\frac{\partial L_{nm}}{\partial \xi_k} = \frac{\partial L_{nk}}{\partial \xi_m} \equiv 0 \quad \text{for} \quad m \neq k.$$

By virtue of diagonality of both $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{\gamma}}$, the consistency condition for the complete time-dependent solution (3.9), with $\boldsymbol{L}_0$ having the structure (3.12), holds at any moment of time.

Now, taking into account the 'matrix' separation of variables (3.12), we can represent the condition $N_{31} = 0$, where N_{31} is given by (3.11*c*), as

$$(\omega + 2\gamma)L_{11}(\xi_1)\bar{L}_{13}(c) + (\omega - 2\gamma)L_{21}(\xi_1)L_{13}(c) + \omega L_{31}(\xi_1)L_{33}(c) = 0\,. \tag{3.13}$$

If this principal requirement following from the system of matrix equations is met by an appropriate choice of elements L_{nm}, matrix $\boldsymbol{L}_0$ can be completed using the relations (2.19). Substituting $\boldsymbol{L}_0$ found from (3.13) into (3.9) we obtain the time-dependent matrix that satisfies all the derived conditions, i.e. the solution to the matrix hydrodynamic equations.

Let us point to a property of the vorticity field common to any solutions in the form (3.9) that can be derived from (3.13). Vorticity is determined by the matrix of the Cauchy invariants $\boldsymbol{M}$ which, for the solutions satisfying (3.13), takes the form $\boldsymbol{M} = \boldsymbol{N} = \mathrm{diag}\{-\mathrm{i}S_3, \mathrm{i}S_3, 0\}$, where

$$S_3 = -2[(\omega + \gamma)|L_{11}|^2 + (\omega - \gamma)|L_{21}|^2 + \omega|L_{31}|^2]. \tag{3.14}$$

We see from (2.9), (2.17) that the real matrix $\boldsymbol{S}$ corresponding to the diagonal $\boldsymbol{M}$ contains only one non-zero invariant S_3. As shown in § 2.4, for this case the vorticity is given by (2.28), and vortex lines follow the coordinate lines of the Lagrangian variable c considered as a curvilinear coordinate in space.

There are several possible ways to solve equation (3.13) by separation of variables, resulting in different classes of fluid motion. We shall discuss them in turn.

3.2. *Planar Ptolemaic flows*

A straightforward way to satisfy (3.13) is to suppose that functions $L_{13}(c)$ and $L_{31}(\xi_1)$ vanish. Then, according to (2.19), so do $L_{23}(\xi_2)$ and $L_{32}(c)$. Under this assumption, all the other elements of $\boldsymbol{L}_0$ appearing in (3.13): $L_{11}(\xi_1)$, $L_{21}(\xi_1)$ and $L_{33}(c)$, can be taken arbitrarily. It is convenient for further analysis to introduce functions $G(\xi_1), F(\xi_2), q(c)$ such that $L_{11}(\xi_1) = G'(\xi_1)$, $L_{21}(\xi_1) = \overline{F'(\xi_2)}$, $L_{33}(c) = q'(c)$, where G and F are analytical functions of their arguments, q is a real-valued function of c (in § 3, the prime denotes differentiation with respect to the argument of a function). After that, we construct the initial complex Jacobi matrix $\boldsymbol{L}_0$ (3.12) using (2.19):

$$\boldsymbol{L}_0 = \begin{bmatrix} G'(\xi_1) & F'(\xi_2) & 0 \\ \overline{F'(\xi_2)} & \overline{G'(\xi_1)} & 0 \\ 0 & 0 & q'(c) \end{bmatrix}. \tag{3.15}$$

Substituting this matrix into (3.9) yields the time-dependent solution to the matrix hydrodynamic equations in complex form. Let us consider the elements of the second row of matrix (3.15) taking into account that ξ_1 and ξ_2 are complex conjugate by the definition (2.15). In conformity with the Riemann–Schwartz symmetry principle (see Hurwitz & Courant 1964), element $\overline{F'(\xi_2)}$ is an analytic function of $\bar{\xi}_2$, that is, of ξ_1:

$\overline{F'(\xi_2)} = \bar{F}'(\bar{\xi}_2) = (\bar{F})'(\xi_1)$ (the notation $\bar{F}$ implies the function defined by $\bar{F}(\zeta) = \overline{F(\bar{\zeta})}$, and as follows directly from that symmetry principle, $\bar{F}' = (\bar{F})'$). By analogy $\overline{G'(\xi_1)}$ is an analytic function of ξ_2. Evidently, the obtained matrix (3.15) fits in the matrix structure (3.12), and the consistency condition (2.16*c*) is satisfied trivially: all the cross-derivatives of the elements of (3.15) vanish.

The particle trajectories corresponding to the obtained matrix solution are found by integrating (3.9) with the matrix $\boldsymbol{L}_0$ (3.15) over the complex Lagrangian variables by analogy with (2.11), which represents the reconstruction of trajectories from the real matrix $\boldsymbol{R}$. For convenience, we shall keep complex notation for particle trajectories: W_1, W_2, Z are the current complex coordinates of a particle as introduced in (2.12), and they depend on complex Lagrangian variables ξ_1, ξ_2, c (2.15) and time. The integration yields

$$W_1 = G(\xi_1)\,\mathrm{e}^{-\mathrm{i}(\omega+\gamma)t} + F(\xi_2)\,\mathrm{e}^{\mathrm{i}(\omega-\gamma)t}, \quad W_2 = \bar{W}_1, \quad Z = q(c). \qquad (3.16a\text{–}c)$$

Here G and F are arbitrary analytical functions of the complex Lagrangian variables, q depends on real Lagrangian variable c, and ω, γ are real constants. Obviously, since Z in (3.16*c*) does not depend on time, this solution is purely two-dimensional, with the Z-component of velocity vanishing and particle motion occurring in the Z = const planes. Exact solutions of this type studied by Abrashkin & Yakubovich (1984) are known as the Ptolemaic flows. This term reflects the following analogy: the particle trajectories in (3.16*a*) represent a combination of two circular revolutions with different amplitudes and different frequencies, thus forming epi- or hypocycloids. It is the type of motion that celestial bodies undergo according to Ptolemaeus' model of the universe. The Ptolemaic solutions (3.16*a*) incorporating two arbitrary analytical functions describe an extensive family of two-dimensional non-stationary rotational motions with vorticity

$$\boldsymbol{\Omega} = -2\left(\frac{|G'|^2 + |F'|^2}{|G'|^2 - |F'|^2}\,\omega + \gamma\right)\{0, 0, 1\} \qquad (3.17)$$

directed normally to the plane of particle trajectories and generally inhomogeneous in space.

A number of exact solutions known in the Lagrangian representation such as the Gerstner waves or Kirchhoff's elliptical vortex are particular cases of this family of solutions. Interest in the Ptolemaic solutions is primarily connected with the fact that they provide an opportunity to study analytically a class of bounded non-stationary vortices surrounded by potential flows. Conventional approaches do not permit obtaining such analytical solutions to this problem except for a few classical particular cases.

Assume that a rotational Ptolemaic motion occurs in the interior of a vortex region corresponding to the unit disc on the plane of the complex Lagrangian variable ξ_1. The general solution (3.16*a*) applies to bounded vortices under the restriction that $\omega = \gamma$, $G(\xi_1) = \xi_1$, and $F(\xi_2)$ is analytical within the unit disc $|\xi_2| \leqslant 1$ and there obeys the condition $|F'| < 1$. In this case (3.16*a*) becomes

$$W_1 = \xi_1\,\mathrm{e}^{-2\mathrm{i}\omega t} + F(\bar{\xi}_1), \qquad (3.18)$$

where ξ_2 is replaced by $\bar{\xi}_1$ in conformity with (2.15). By substituting $\xi_1 = \exp(\mathrm{i}\varphi)$ into (3.18) we straightforwardly obtain the parametric expression for the vortex boundary corresponding to the unit circle $|\xi_1| = 1$:

$$W_1 = \mathrm{e}^{\mathrm{i}(\varphi-2\omega t)} + F(\mathrm{e}^{-\mathrm{i}\varphi}), \qquad (3.19)$$

where φ varies from 0 to 2π. It is seen from the above expression that the shape of the vortex region is repeated periodically with period π/ω. For the complex coordinates (2.12) that we have adopted, it is natural to define a complex velocity as $V_2 = 2^{-1/2}(V_x - \mathrm{i}V_y) = \partial \bar{W}_1/\partial t$, V_x and V_y being X- and Y-components of the velocity (partial time differentiation implies fixed Lagrangian variables). The velocity is easily obtained from (3.19), and the vorticity (3.17) becomes

$$\boldsymbol{\Omega} = -\frac{4\,\omega}{1-|F'|^2}\,\{0,0,1\} \tag{3.20}$$

and remains bounded inside the vortex due to the assumption $|F'| < 1$.

As shown by Abrashkin & Yakubovich (1984), any bounded Ptolemaic motion matches with a potential flow under the continuity condition across the interface for all velocity components. The potential flow is constructed in parametric form. For the exterior of the vortex (3.19), complex coordinate W_1 and the corresponding velocity V_2 can be expressed as functions of an auxiliary complex parameter η varying outside the unit circle ($|\eta| \geqslant 1$):

$$W_1 = \eta\,\mathrm{e}^{-2\mathrm{i}\omega t} + F(\eta^{-1}), \quad V_2 = 2\mathrm{i}\,\omega\,\mathrm{e}^{2\mathrm{i}\omega t}\eta^{-1}. \tag{3.21a,b}$$

Note that here η is a formal parameter which has nothing to do with the Lagrangian variables in the outer region. Since the coordinate and velocity in (3.21a,b) depend on the same parameter, they define the function $V_2 = V_2(W_1, t)$. It is known (see Lamb 1932) that such a functional dependence is necessary and sufficient for the motion to be potential. Direct verification shows that the coordinate and velocity at the vortex boundary found from (3.21a,b) by means of the substitution $\eta = \exp(\mathrm{i}\varphi)$, $0 \leqslant \varphi < 2\pi$ coincide with the solution for the vortex interior when ξ_1 tends to $\exp(\mathrm{i}\varphi)$ in (3.18). It can also be shown using (3.21a,b) that under the condition $|F'| < 1$ when $|\xi_1| \leqslant 1$, the velocity in the outer region has no singularities as a function of coordinate $V_2 = V_2(W_1, t)$. As the distance from the vortex increases, the velocity (3.21b) tends asymptotically to that of the point vortex with circulation $-4\pi\omega$: $V_2 \sim 2\mathrm{i}\omega/W_1$.

Thus, expressions (3.18) and (3.21a,b) depending on arbitrary analytic function $F(\bar{\xi}_1)$ give a complete description of the Ptolemaic vortices. An example of the evolution of the shape of vortex boundary (3.19) for

$$F(\xi_1) = \alpha\left[(\xi_1-\zeta_1)^{-3} + (\xi_1-\zeta_2)^{-3}\right], \tag{3.22}$$

α, ζ_1, ζ_2 being constants ($|\zeta_{1,2}| > 1$), is shown in figure 1(a–f). Along with non-stationary solutions, the Ptolemaic family includes steadily rotating vortices of hypocycloidal shape which correspond to the power function $F(\xi_1) = \xi_1^n$ for an integer $n \geqslant 2$. Both types of vortices were first described by Abrashkin & Yakubovich (1984).

The planar Ptolemaic flows are obtained through the simplest case of separation of variables in (3.13). But there exist other possible ways to satisfy this equation resulting in other types of solutions of the form (3.6). We now consider solutions which are essentially three-dimensional, unlike the Ptolemaic ones.

3.3. *Flows with curvilinear vortex lines*

We begin with one of the cases when the separation of variables in (3.13) for elements of $\boldsymbol{L}_0$ is non-trivial. Assume that $L_{31}(\xi_1) = 0$, then (3.13) yields

$$\frac{(2\gamma+\omega)L_{11}(\xi_1)}{(2\gamma-\omega)L_{21}(\xi_1)} = \frac{L_{13}(c)}{\bar{L}_{13}(c)} = \text{const.} \tag{3.23}$$

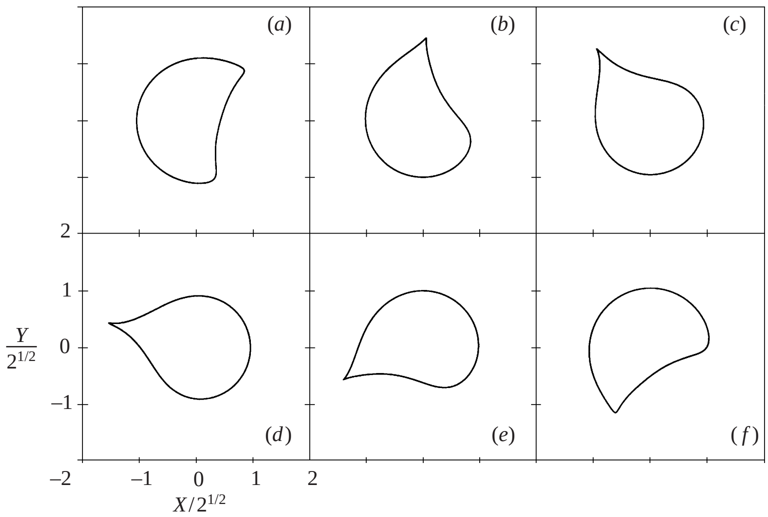

FIGURE 1. Shapes of the Ptolemaic vortex (3.19) for function F specified by (3.22), where $\alpha = 0.83$, $\zeta_1 = 2.2$, $\zeta_2 = 2.09 + 0.68\mathrm{i}$, for a sequence of times: (*a*) $\omega t = 0$; (*b*) $\omega t = \pi/12$; (*c*) $\omega t = 2\pi/12$; (*d*) $\omega t = 3\pi/12$; (*e*) $\omega t = 4\pi/12$; (*f*) $\omega t = 5\pi/12$.

Here, the absolute value of the constant of separation of variables must be equal to unity, so it can be rewritten via a new constant v as $\mathrm{e}^{2\mathrm{i}v}$. $L_{33}(c)$ does not appear in (3.23) and thus can be an arbitrary function. For convenience of further analysis, let us represent L_{11}, L_{13}, L_{33} as $L_{11} = (2-\omega/\gamma)F'(\xi_1)\,\mathrm{e}^{\mathrm{i}v}$, $L_{13} = h'(c)\,\mathrm{e}^{\mathrm{i}v}$, $L_{33} = q'(c)$, where $F(\xi_1)$ is an arbitrary analytical function and h, q are arbitrary real-valued functions of c. Then (3.23) is satisfied, and all the other elements L_{ij} are found from (2.19). Matrix $\boldsymbol{L}_0$ takes the form

$$\boldsymbol{L}_0 = \begin{bmatrix} (2-\omega/\gamma)F'(\xi_1)\,\mathrm{e}^{\mathrm{i}v} & (2+\omega/\gamma)(\bar{F})'(\xi_2)\,\mathrm{e}^{\mathrm{i}v} & h'(c)\,\mathrm{e}^{\mathrm{i}v} \\ (2+\omega/\gamma)F'(\xi_1)\,\mathrm{e}^{-\mathrm{i}v} & (2-\omega/\gamma)(\bar{F})'(\xi_2)\,\mathrm{e}^{-\mathrm{i}v} & h'(c)\,\mathrm{e}^{-\mathrm{i}v} \\ 0 & 0 & q'(c) \end{bmatrix} \tag{3.24}$$

and satisfies all the requirements that we have formulated. The meaning of the notation $(\bar{F})'$ is explained in the first paragraph of § 3.2. To obtain a complete time-dependent matrix solution, (3.24) should be substituted into (3.9). Integration of the resulting matrix $\boldsymbol{L}$ over Lagrangian variables is straightforward and for particle trajectories yields

$$W_1 = \left[(2-\omega/\gamma)F(\xi_1)\,\mathrm{e}^{-\mathrm{i}\omega t} + (2+\omega/\gamma)\bar{F}(\xi_2)\,\mathrm{e}^{\mathrm{i}\omega t} + h(c)\right]\mathrm{e}^{(v-\mathrm{i}\gamma t)}, \tag{3.25a}$$

$$W_2 = \bar{W}_1, \quad Z = q(c), \tag{3.25b,c}$$

where complex notation for coordinates (2.12) holds. As seen from (3.25*a*), the constant v does not influence the structure of solutions and their dependence on time: the factor $\mathrm{e}^{\mathrm{i}v}$ implies only rotation of the motion as a whole at the fixed angle v about the Z-axis, and so we set $v = 0$ without losing generality. The solution (3.25*a–c*) contains two arbitrary functions of the real Lagrangian variable c and one arbitrary function of complex variables ξ_1, ξ_2. However, as mentioned in § 2.1, the system of the Lagrangian variables can be rearranged arbitrarily for the sake of convenience. For instance, one of the arbitrary functions for each Lagrangian variable can be taken as

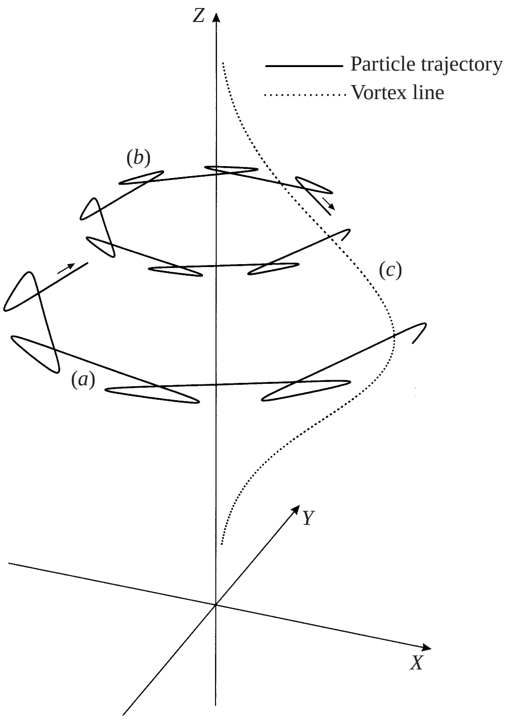

FIGURE 2. Particle trajectories (3.26a,b) for $h(c) = 4\exp[-(c-7)^2/9]$, $\omega = 8.5$, $\gamma = 1$, and (a) $\xi_1 = 0.1$, $c = 7$; (b) $\xi_1 = 0.06$, $c = 9$ and the corresponding vortex line (c) for $\xi_1 = 0$, $t = 0$ (for discussion of vortex lines see below).

the new Lagrangian variable, say, $\tilde{c} = q(c)$, $\tilde{\xi}_1 = F(\xi_1)$, and the remaining arbitrary functions can be expressed in terms of the new variables: $\tilde{h}(\tilde{c}) = h(q^{-1}(\tilde{c}))$. It means that the solution (3.25a–c) is in no way restricted by reduction to

$$W_1 = \left[(2-\omega/\gamma)\xi_1 \mathrm{e}^{-\mathrm{i}\omega t} + (2+\omega/\gamma)\bar{\xi}_1 \mathrm{e}^{\mathrm{i}\omega t} + h(c)\right] \mathrm{e}^{-\mathrm{i}\gamma t}, \quad Z = c\,. \qquad (3.26a,b)$$

In a general case, the particle motion (3.26a,b) occurs along the trajectories representing a sum of three circular revolutions with different amplitudes and frequencies $\omega+\gamma, \omega-\gamma, \gamma$ in the $Z =$ const planes. In other words, a particle travels in the $Z =$ const plane about an ellipse, the centre of which moves over a circle as shown in figure 2. For incommensurable ω and γ, the trajectories are non-closed and quasi-periodic. Although they lie in parallel planes, the motion is three-dimensional, since due to the term $h(c)$ in (3.26a) there exists a shear between plane layers. The function $h(c)$ is supposed to be single-valued so that $1/h'(c) \neq 0$. If $h(c) = 0$, solution (3.26a,b) reduces to a particular case of planar Ptolemaic motions (3.16a–c).

To obtain the vorticity for this type of solution, we find the only non-zero Cauchy invariant in conformity with (3.14) and substitute it into the general expression (2.28):

$$\boldsymbol{\Omega} = \frac{\omega^2}{2\gamma}\{2^{1/2}\,h'(c)\cos\gamma t,\ -2^{1/2}\,h'(c)\sin\gamma t,\ 1\}. \qquad (3.27)$$

This is at the same time the Eulerian representation of vorticity since $c = Z$. At any point in the flow, the vorticity vector precesses around the direction of the Z-axis with frequency γ. As proved in § 3.1, § 2.4, parametric expressions for the vortex lines follow immediately from (3.26a,b), if ξ_1 is set constant while c is allowed to change.

According to (3.26a, b), the vortex lines are flat curves of identical shape described by the function $h(c)$ (see figure 2). At any moment of time, they lie in parallel planes and are distinguished only by the starting point, say, corresponding to $c = 0$, and can be superimposed by a translation in the $Z = \text{const}$ plane.

It is noteworthy that the particle trajectories (3.26a, b) and related velocities allow us to eliminate the Lagrangian variables and obtain the explicit expression for the Eulerian velocity field in complex form:

$$W_{1t} = \frac{\mathrm{i}\omega^2}{4\gamma}\left[W_1 + \left(1 - \frac{4\gamma^2}{\omega^2}\right)\bar{W}_1\,\mathrm{e}^{-2\mathrm{i}\gamma t} - 2\,h(Z)\,\mathrm{e}^{-\mathrm{i}\gamma t}\right], \quad Z_t = 0.$$

Here, the first term of W_{1t} represents a stationary circular motion generating the Z-component of vorticity, the second term contains a non-stationary two-dimensional irrotational strain, and the third term characterizing the vertical shear is responsible for the oscillating vorticity components in the (X, Y)-plane.

A complete study of the motion (3.26a, b) implies considering relevant boundary problems. Here we restrict ourselves to determining the shape of rigid boundaries which may confine the above-mentioned ideal-fluid motion. Let us find an envelope surface of particle trajectories in the frame of reference rotating about the Z-axis with frequency γ. This envelope surface can be then replaced by a solid boundary at which the impermeable boundary conditions are satisfied. Thus, coming back to the laboratory frame of reference we have the necessary condition satisfied for a rotating solid body. Analysis of (3.26a, b) shows that the appropriate shapes constitute an extensive family of rotating surfaces distinguished by elliptical cross-sections in the (X, Y)-planes with a certain eccentricity and orientation of principal axes. A parametric equation of such surfaces is derived in the form

$$W_1 = \left\{\left[(2 - \omega/\gamma)\,\mathrm{e}^{-\mathrm{i}\varphi} + (2 + \omega/\gamma)\,\mathrm{e}^{\mathrm{i}\varphi}\right] r(Z) + h(Z)\right\}\mathrm{e}^{-\mathrm{i}\gamma t},$$

where $r(Z)$ is an arbitrary function which characterizes the dimension of the cross-section in the $Z = \text{const}$ plane, and φ is an angular parameter varying within $0 \leqslant \varphi < 2\pi$.

We shall consider more complex problems of matching with a potential flow later in §4.

3.4. *Flows with rectilinear vortex lines*

In order to study other cases of the solutions contained in (3.6), (3.9) let us discuss one more way to separate variables in (3.13). Assume that L_{13} is real, then (3.13) can be satisfied by

$$\frac{L_{31}(\xi_1)}{(\omega + 2\gamma)L_{11}(\xi_1) + (\omega - 2\gamma)L_{21}(\xi_1)} = -\frac{L_{13}(c)}{\omega L_{33}(c)} = \lambda, \tag{3.28}$$

where λ is a real constant having the dimension of time. Separation of variables in (3.13) is also possible for complex L_{13} when $L_{13} = f(c)\mathrm{e}^{\mathrm{i}\nu}$ with $f(c)$ real-valued and ν constant (independent of c). However, it can be shown by analogy with (3.25) that taking account of constant phase shift due to the term $\mathrm{e}^{\mathrm{i}\nu}$ does not allow any considerable extension of the resulting solutions, and therefore let us assume $\mathrm{Im}L_{13} = 0$ for simplicity. We see from (3.28) that two functions out of $L_{11}(\xi_1)$, $L_{21}(\xi_1)$, $L_{31}(\xi_1)$ can be taken arbitrarily and the third one must be their linear combination. Let us choose $L_{11} = G'(\xi_1)$, $L_{21} = F'(\xi_1)$, $L_{33} = q'(c)$. We then obtain using (2.19) the

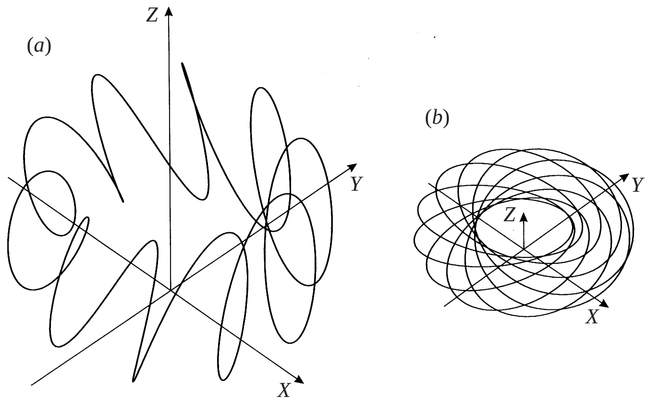

FIGURE 3. Particle trajectories (3.30a–c) for $F(\xi_1) = \alpha\,\xi_1$, $\alpha = 45 + 23\mathrm{i}$, $\xi_1 = 0.1 + 0.1\mathrm{i}$, $\lambda = -0.04$, $\omega = 10$, $\gamma = 1$. (a) $c = 75$; (b) $c = 7.5$.

following structure of matrix $\boldsymbol{L}_0$:

$$\boldsymbol{L}_0 = \begin{bmatrix} G'(\xi_1) & (\bar{F})'(\xi_2) & -\omega\lambda\, q'(c) \\ F'(\xi_1) & (\bar{G})'(\xi_2) & -\omega\lambda\, q'(c) \\ \lambda[(\omega+2\gamma)G' + (\omega-2\gamma)F'] & \lambda[(\omega+2\gamma)(\bar{G})' + (\omega-2\gamma)(\bar{F})'] & q'(c) \end{bmatrix}. \tag{3.29}$$

The meaning of notation $(\bar{F})', (\bar{G})'$ is the same as introduced in the first paragraph of § 3.2. Particle trajectories are derived by integration of (3.6) with the obtained $\boldsymbol{L}_0$ over the Lagrangian variables. Proceeding from the same argumentation that we employed to reduce (3.25a–c) to (3.26a,b), setting $G(\xi_1) = \xi_1$, $q(c) = c$ does not restrict the considered family of solutions. The resulting particle trajectories take the form

$$W_1 = \left[\xi_1 \mathrm{e}^{-\mathrm{i}\omega t} + \bar{F}(\xi_2)\mathrm{e}^{\mathrm{i}\omega t} - \omega\lambda c\right]\mathrm{e}^{-\mathrm{i}\gamma t}, \quad W_2 = \bar{W}_1, \tag{3.30a,b}$$

$$Z = 2\lambda\,\mathrm{Re}\left\{[(\omega+2\gamma)\xi_1 + (\omega-2\gamma)F(\xi_1)]\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} + c \tag{3.30c}$$

with only one arbitrary function F that really influences the structure of the flows. The trajectories (3.30a–c) contain four time-dependent harmonic terms with frequencies $\omega+\gamma, \omega-\gamma, \gamma, \omega$. In space, they represent windings on toroidal surfaces, which are formed by rotation about the Z-axis of ellipses having different orientations as shown in figure 3(a,b). For incommensurable frequencies ω and γ these windings are quasi-periodic, and each trajectory fills in the entire surface of the relevant torus. The vorticity of such flows is obtained from (2.28) and (3.14):

$$\boldsymbol{\Omega} = \frac{S_3}{\det \boldsymbol{L}_0}\{-2^{1/2}\,\omega\lambda\cos\gamma t,\ 2^{1/2}\,\omega\lambda\sin\gamma t,\ 1\}, \tag{3.31a}$$

where

$$S_3 = -2[\omega+\gamma+(\omega-\gamma)|F'|^2 + \omega\lambda^2|\omega+2\gamma+(\omega-2\gamma)F'|^2], \tag{3.31b}$$

$$\det \boldsymbol{L}_0 = (1+2\omega^2\lambda^2)(1-|F'|^2) + 4\omega\gamma\lambda^2|1-F'|^2. \tag{3.31c}$$

As proved in § 3.1, § 2.4, formulae (3.30a–c) with fixed ξ_1 and varying c describe parametrically a vortex line which is at the same time the material line containing particles with given ξ_1. That is the case when vortex lines are rectilinear and remain inclined at a constant angle to the Z-direction in the course of their motion, while their projections on the (X, Y)-plane rotate with frequency γ. Unlike the vorticity of motions (3.26*a,b*) given by (3.27), the vorticity (3.31*a–c*) is, generally, a function of the complex Lagrangian variables.

If $\lambda = 0$, the motion (3.30) becomes a plane Ptolemaic flow (3.16*a–c*) with vorticity in the Z-direction. Therefore, along with the motions (3.25*a–c*), the flows (3.30*a–c*) may be considered as three-dimensional generalizations of the Ptolemaic flows.

Consider now elementary boundary problems for the motions (3.30*a–c*). If the function $F(\xi_1)$ is linear with respect to the complex Lagrangian variable: $F = \alpha\xi_1$, $\alpha = \text{const}$, then the flow can satisfy the impermeability conditions at the rotating rigid boundaries described parametrically by

$$W_1 = \left[\left(e^{-i\varphi} + \bar{\alpha}\, e^{i\varphi}\right) r(c) - \omega\lambda c\right] e^{-i\gamma t}\,, \tag{3.32a}$$

$$Z = 2\lambda\, \mathrm{Re}\{[\omega + 2\gamma + \alpha(\omega - 2\gamma)]\, e^{i\varphi}\} r(c) + c, \tag{3.32b}$$

where $r(c)$ is an arbitrary real-valued function, φ $(0 \leqslant \varphi < 2\pi)$ and c are parameters.

Let us now demonstrate how one of the classical solutions for a rotating liquid mass, namely, a subclass of Riemann's ellipsoids (see Chandrasekhar 1969), can be deduced from the free-surface boundary problem for the motions (3.30*a–c*). The pressure field of the flows (3.30*a–c*) is determined by substituting these solutions into the original set of Lagrangian equations (2.2) with subsequent integration of the pressure gradient over the Lagrangian variables. The manipulation yields

$$\begin{aligned}\frac{p}{\rho} &= \omega^2\gamma^2\lambda^2c^2 + (\omega+\gamma)^2|\xi_1|^2 + (\omega-\gamma)^2|F|^2 + \lambda^2\omega^2|(\omega+2\gamma)\xi_1 + (\omega-2\gamma)F|^2 \\ &\quad +2\,\mathrm{Re}\{-\lambda\omega\gamma^2c\,(\xi_1+F)\,e^{-i\omega t} + [\{(\omega+\gamma)^2 + \lambda^2\omega^2(\omega^2-4\gamma^2)\}\xi_1F \\ &\quad -4\gamma\omega\int F\,d\xi_1 + \tfrac{1}{2}\lambda^2\omega^2\{(\omega+2\gamma)^2\xi_1^2 + (\omega-2\gamma)^2F^2\}]e^{-2i\omega t}\}.\end{aligned} \tag{3.33}$$

Assume that $F = -\xi_1$. It is then seen from (3.33) that for a certain relationship of the parameters, pressure may become independent of time, i.e. change into a function of the Lagrangian variables only:

$$p = \rho\,(\omega^2+\gamma^2)(4\,|\,\xi_1|^2 + \tfrac{1}{8}c^2) \quad \text{for} \quad \lambda^2 = (\omega^2+\gamma^2)/8\gamma^2\omega^2. \tag{3.34a, b}$$

Constancy of the pressure (3.34*a*) is the boundary condition on the free surface of a liquid mass involved in the motion (3.30*a–c*) with the restriction (3.34*b*) on the parameters. The equation of the free surface in terms of the Lagrangian variables follows immediately: $|\,\xi_1|^2 + \frac{1}{32}c^2 = \text{const}$. The corresponding equation of ellipsoids in space is obtained from (3.30*a–c*). It can be shown that both vorticity and angular velocity vectors lie in a principal plane of these ellipsoids, which is the distinctive feature of a certain subclass of Riemann's ellipsoids according to Chandrasekhar (1969).

3.5. *Special case of flows with rectilinear vortex lines*

Let us analyse one more type of flow with rectilinear vortex lines that is derived from the matrix expression (3.6). It is distinguished from the motions studied in § 3.4 by the fact that its vortex lines are strictly perpendicular to the axis of rotation Z. This case does not fall under the procedure of variable separation (3.28) (otherwise it would correspond to $\lambda = \infty$) and should be considered separately.

Suppose that in (3.13) $L_{33} = 0$, then the function $L_{31}(\xi_1)$ is arbitrary, while elements L_{11}, L_{21} and L_{13} are related by (3.23). It can be shown by analogy with the analysis of (3.24), (3.25a–c) in § 3.3 that taking L_{13} as a real-valued function is sufficient for a comprehensive study of the solutions of this type. Assume that L_{11}, L_{21} are expressed in terms of an arbitrary analytical function $G(\xi_1)$ as $L_{11} = (2-\omega/\gamma)G'(\xi_1)$, $L_{21} = (2+\omega/\gamma)G'(\xi_1)$, then (3.23) is satisfied provided that $\operatorname{Im} L_{13} = 0$. Introduce new arbitrary functions for L_{31}, L_{13} by $L_{31} = F'(\xi_1), L_{13} = q'(c)$, then the general form of matrix $\boldsymbol{L}_0$ is constructed using (2.19) as follows:

$$\boldsymbol{L}_0 = \begin{bmatrix} (2-\omega/\gamma)G'(\xi_1) & (2+\omega/\gamma)(\bar{G})'(\xi_2) & q'(c) \\ (2+\omega/\gamma)G'(\xi_1) & (2-\omega/\gamma)(\bar{G})'(\xi_2) & q'(c) \\ F'(\xi_1) & (\bar{F})'(\xi_2) & 0 \end{bmatrix}, \tag{3.35}$$

where $\bar{G}, \bar{F}$ are 'conjugate' functions to G, F defined via the Riemann–Schwartz symmetry principle, e.g. $\bar{G}(\zeta) = \overline{G(\bar{\zeta})}$ (see the discussion in § 3.2 after (3.15)). As in the derivation of (3.26a, b) and (3.30a–c), keeping unnecessary arbitrary functions in the further development does not make sense because of the parametric character of the Lagrangian solutions. For simplicity we set $G(\xi_1) = \xi_1$ and $q(c) = c$ without losing generality. Then we obtain particle trajectories by integration over the Lagrangian variables:

$$W_1 = [(2-\omega/\gamma)\xi_1 \mathrm{e}^{-\mathrm{i}\omega t} + (2+\omega/\gamma)\bar{\xi}_1 \mathrm{e}^{\mathrm{i}\omega t} + c]\,\mathrm{e}^{-\mathrm{i}\gamma t}, \tag{3.36a}$$

$$Z = 2\operatorname{Re}\left\{F(\xi_1)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}. \tag{3.36b}$$

The vorticity vector of such a flow

$$\boldsymbol{\Omega} = \frac{2\omega^2 + \gamma^2|F'|^2}{2\gamma \operatorname{Re} F'}\{-2^{1/2}\cos\gamma t,\ 2^{1/2}\sin\gamma t,\ 0\} \tag{3.37}$$

lies in the $Z = \text{const}$ planes and rotates with frequency $-\gamma$.

Let us finally return to the general matrix expression (3.6) embracing all the family of solutions with precessing vorticity and specify what type of flow corresponds to its particular case (3.4) where matrix $\hat{\boldsymbol{\gamma}}$ does not appear: $\boldsymbol{R} = \boldsymbol{R}_0 \exp(\hat{\boldsymbol{\omega}} t)$. Obviously, the precession of vorticity with frequency γ vanishes in this case and the motion reduces to the planar Ptolemaic flow (3.16a–c) in which $\gamma = 0$. This can be demonstrated by setting $\gamma = 0$ in the three-dimensional solutions (3.30a–c) and (3.36a, b) with an appropriate rotation of the reference frame so as to make the new Z-axis parallel to vorticity.

4. Precession of cylindrical vortices in irrotational strain

The three-dimensional motions that have been studied in § 3 above are unbounded and rotational at any point in space. Obviously, bounded vortex structures within an irrotational exterior motion or rotational flows decaying at infinity are more realistic physically. As seen from (3.26a, b), (3.30a–c), (3.36a, b), none of these solutions can describe a rotational motion vanishing at infinity. On the other hand, it was shown by Abrashkin & Yakubovich (1984) that matching with a potential flow is possible for any two-dimensional Ptolemaic motion (3.16a–c). Since our solutions (3.26a, b), (3.30a–c) are a generalization of the planar Ptolemaic motions, it is natural to question whether their matching with a potential flow may take place (when using the term 'matching' we imply that all the components of flow velocities agree across the interface).

It is known that if there are no rigid boundaries, a region of rotational motion can be bounded only by surfaces which are composed of vortex lines (see e.g. Saffman 1992, § 1.4). For all the solutions in the form (3.6) that we have studied, vortex lines begin and end at infinity, and therefore rotational motion can be localized only in the directions transverse to vorticity. Thus we can formulate the problem of matching with an irrotational flow for a vortex tube of finite cross-section of unbounded motions (3.26*a*, *b*), (3.30*a*–*c*) or (3.36*a*, *b*).

Let us focus on the detailed analysis of solution (3.30*a*–*c*) with the linear function $F = \alpha\xi_1$, which reduces to

$$W_1 = \left(\xi_1\,\mathrm{e}^{-\mathrm{i}\omega t} + \alpha\,\bar{\xi}_1\,\mathrm{e}^{\mathrm{i}\omega t} + \beta c\right)\mathrm{e}^{-\mathrm{i}\gamma t}, \quad W_2 = \bar{W}_1, \quad Z = 2\sigma\mathrm{Re}\left(\xi_1\mathrm{e}^{-\mathrm{i}\omega t}\right) + c, \quad (4.1a\text{–}c)$$

where α, $\beta = -\omega\lambda$ and $\sigma = \lambda[(\omega+2\gamma)+\alpha(\omega-2\gamma)]$ are real constants, $\alpha \neq 1$. According to (3.31*a*–*c*), the vorticity of such a motion has a constant magnitude and precesses about the Z-direction with frequency γ:

$$\boldsymbol{\Omega} = 2\frac{\omega\left[\omega+\gamma+\alpha^2(\omega-\gamma)\right] + 4\beta^2\left[\omega+2\gamma+\alpha(\omega-2\gamma)\right]^2}{\omega(1+2\beta^2)(1-\alpha^2) + 4\gamma\beta^2(1-\alpha)^2}\begin{bmatrix} -2^{1/2}\beta\cos\gamma t \\ 2^{1/2}\beta\sin\gamma t \\ -1 \end{bmatrix}. \quad (4.2)$$

Common to the entire family of solutions (3.9) is the fact that their vortex lines are identical to the coordinate lines of the Lagrangian variable c (see § 3.1, § 2.4). In particular, the parametric expression for vortex lines is obtained from (4.1*a*–*c*) immediately by fixing ξ_1 and letting c change along the lines. The vortex lines of (4.1*a*–*c*) are evidently rectilinear and follow the direction of the vorticity (4.2). Consider the vortex lines that correspond to $\xi_1 = \exp(\mathrm{i}\varphi), 0 \leqslant \varphi < 2\pi$. They represent the elements of the cylindrical surface that confine a vortex tube within the flow (4.1*a*–*c*). Substituting $\xi_1 = \exp(\mathrm{i}\varphi)$ into (4.1*a*–*c*) we obtain the expression for this surface

$$W_1 = [\mathrm{e}^{\mathrm{i}(\varphi-\omega t)} + \alpha\mathrm{e}^{-\mathrm{i}(\varphi-\omega t)} + \beta c]\mathrm{e}^{-\mathrm{i}\gamma t}, \quad Z = 2\sigma\cos(\varphi-\omega t) + c \qquad (4.3a, b)$$

in two parameters φ and c, where $0 \leqslant \varphi < 2\pi$, $-\infty < c < \infty$. It can be seen that (4.3*a*, *b*) describes a cylinder of elliptic cross-section that precesses about the Z-axis with frequency γ as shown schematically in figure 4, where the axis of the cylinder is the vortex line corresponding to $\xi_1 = 0$ in (4.1*a*–*c*). A more explicit representation for the shape of this vortex tube will be derived below in this section.

This section is aimed at studying whether the precessing vortex tube (4.3*a*, *b*) can be an isolated vortex surrounded by an irrotational fluid, provided that all three velocity components are continuous across the vortex boundary. There is no generally applicable technique for constructing a potential flow that matches a three-dimensional, non-stationary rotational motion in a bounded region. In fact, the problem of matching is essentially self-consistent and requires a coordinated solution for both potential and rotational motions. For a given fragment of an unbounded rotational motion, the continuity of both normal and tangential velocities across the interface constitutes a boundary problem for Laplace's equation in the outer region where both Dirichlet and Neumann conditions must be satisfied simultaneously on a time-dependent surface.

The problem of potential continuation of the rotational motion (4.1*a*–*c*) into the exterior of the vortex tube (4.3*a*, *b*) will be treated according to the following scheme: on passing to the reference frame moving in accordance with the precession of the vortex tube, the Eulerian velocity field is decomposed into two partial two-dimensional fields, one of them being potential and the other being similar to the velocity field inside Kirchhoff's vortex. The potential field is extended into the exterior trivially,

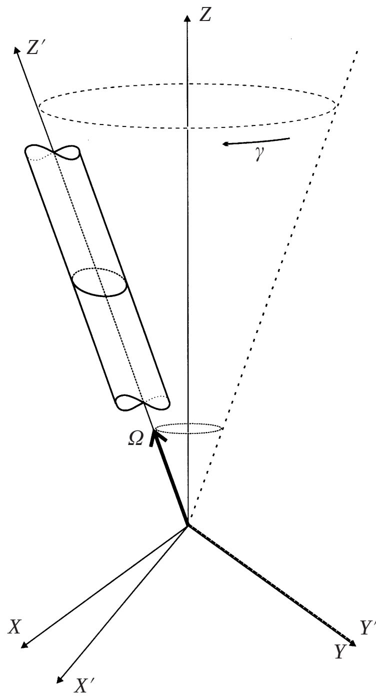

FIGURE 4. Sketch of a section of the precessing vortex tube and the moving coordinate system X', Y', Z' at $t = 0$.

and for the planar rotational component we apply the technique of parametric continuation described in § 3.2. The resulting potential motion outside the vortex proves to be three-dimensional and non-stationary.

Let us proceed to a detailed description of the method of construction of the outer potential field. In the matrix notation the inner rotational flow (4.1a–c) takes the form

$$\boldsymbol{W} = \boldsymbol{L}\,\boldsymbol{\xi}, \quad \boldsymbol{L} = \begin{bmatrix} e^{-i(\omega+\gamma)t} & \alpha\, e^{i(\omega-\gamma)t} & \beta\, e^{-i\gamma t} \\ \alpha\, e^{-i(\omega-\gamma)t} & e^{i(\omega+\gamma)t} & \beta\, e^{i\gamma t} \\ \sigma\, e^{-i\omega t} & \sigma\, e^{i\omega t} & 1 \end{bmatrix}, \tag{4.4a,b}$$

where $\boldsymbol{L}$ does not depend on the Lagrangian variables and is a function of time only. Complex velocity components

$$\boldsymbol{V}_c = \{V_1, V_2, V_3\} = \boldsymbol{T}\,\boldsymbol{V}, \tag{4.5}$$

where $\boldsymbol{T}$ is the same as in (2.13), $\boldsymbol{V} = \{V_x, V_y, V_z\} = \boldsymbol{X}_t$ is the real velocity, are obtained as the time derivatives of the particle positions with fixed Lagrangian variables:

$$\boldsymbol{V}_c = \boldsymbol{W}_t = \boldsymbol{L}_t\,\boldsymbol{\xi}. \tag{4.6}$$

Since the dependence of velocity on $\boldsymbol{\xi}$ is linear, the same type of dependence on coordinates in the Eulerian representation follows after eliminating the Lagrangian variables from (4.4a), (4.6):

$$\boldsymbol{V}_c = \boldsymbol{L}_t\,\boldsymbol{L}^{-1}\boldsymbol{W}. \tag{4.7}$$

Matching of the rotational flow inside the vortex with the outer potential motion is performed in the moving frame of reference $X'Y'Z'$ introduced so that the axis of the vortex serves as the Z'-axis of the new coordinate system as shown in figure 4. The quantities related to the moving reference frame will be marked hereinafter by a prime, unlike § 3, where the prime denotes differentiation. The orthogonal transformation to the precessing reference frame

$$\boldsymbol{X}' = \boldsymbol{P}\boldsymbol{X}, \quad \boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}^{-1} \tag{4.8a, b}$$

must satisfy the condition that a unit vector collinear to the vorticity (4.2) has the components $\{0, 0, 1\}$ in the new reference frame. Matrix $\boldsymbol{P}$ is chosen in the following form:

$$\boldsymbol{P} = \varkappa \begin{bmatrix} \cos\gamma t & -\sin\gamma t & -2^{-1/2}\beta \\ \varkappa^{-1}\sin\gamma t & \varkappa^{-1}\cos\gamma t & 0 \\ 2^{-1/2}\beta\,\cos\gamma t & -2^{-1/2}\beta\,\sin\gamma t & 1 \end{bmatrix}, \quad \text{where } \varkappa = (1+2\beta^2)^{-1/2}. \tag{4.9}$$

New complex coordinates $\boldsymbol{W}' = \{W_1', W_2', W_3'\}$ are introduced by analogy with (2.12), (2.13):

$$W_1' = 2^{-1/2}(X' + \mathrm{i}Y'),\ W_2' = 2^{-1/2}(X' - \mathrm{i}Y'),\ W_3' = Z', \quad \boldsymbol{W}' = \boldsymbol{T}\boldsymbol{X}'. \tag{4.10}$$

They are related to the complex Lagrangian variables $\boldsymbol{\xi}$ via the modified Jacobi matrix $\boldsymbol{L}'$:

$$\boldsymbol{W}' = \boldsymbol{L}'\boldsymbol{\xi}, \quad \text{where} \quad \boldsymbol{L}' = \boldsymbol{T}\boldsymbol{P}\boldsymbol{T}^*\boldsymbol{L}. \tag{4.11}$$

The elements of $\boldsymbol{L}'$ are derived by means of elementary but cumbersome manipulations reflecting the change of the reference frame that involve the matrices $\boldsymbol{T}$, $\boldsymbol{L}$, and $\boldsymbol{P}$ given in (2.13), (4.4*b*), and (4.9), respectively:

$$\left.\begin{aligned} &L_{11}' = \overline{L'}_{22} = l_1\,\mathrm{e}^{-\mathrm{i}\omega t}, && l_1 = \tfrac{1}{2}[\varkappa(\alpha+1-2\beta\sigma) - \alpha + 1], \\ &L_{12}' = \overline{L'}_{21} = l_2\,\mathrm{e}^{\mathrm{i}\omega t}, && l_2 = \tfrac{1}{2}[\varkappa(\alpha+1-2\beta\sigma) + \alpha - 1], \\ &L_{31}' = \overline{L'}_{32} = l_3\,\mathrm{e}^{-\mathrm{i}\omega t}, && l_3 = 2\varkappa\beta\gamma(\alpha-1), \\ &L_{13}' = L_{23}' = 0, && L_{33}' = \varkappa^{-1}. \end{aligned}\right\} \tag{4.12}$$

The parametric equation of the vortex boundary in the new complex coordinates is obtained by the substitution of $\xi_1 = \exp[\mathrm{i}(\varphi + \omega t)]$, $\xi_2 = \exp[-\mathrm{i}(\varphi + \omega t)]$ into (4.11):

$$W_1' = l_1\,\mathrm{e}^{\mathrm{i}\varphi} + l_2\,\mathrm{e}^{-\mathrm{i}\varphi}, \quad W_2' = \overline{W_1'}, \tag{4.13a, b}$$

$$Z' = 2\,l_3\cos\varphi + c/\varkappa, \tag{4.13c}$$

where the real constants l_1, l_2, l_3 are given in (4.12), φ and c are parameters: $0 \leqslant \varphi < 2\pi$, $-\infty < c < \infty$. Now the shape of the cross-section of the vortex tube through the (X', Y')-plane is described explicitly by (4.13*a*) and proves to be elliptical and steady in the moving frame of reference. The elements of the elliptic cylinder corresponding to the change of c while φ is constant are, according to (4.13*a–c*), parallel to the Z'-axis as was to be expected (see figure 4).

The velocity in the moving frame of reference, $\boldsymbol{X}_t'$, obtained from (4.8*a*) consists of two terms: $\boldsymbol{P}\boldsymbol{X}_t$ that represents projections of velocity $\boldsymbol{V} = \boldsymbol{X}_t$ on the axes of the precessing coordinate system, and $\boldsymbol{P}_t\boldsymbol{X}$ that accounts for the relative motion of the two reference frames. We retain hereafter only the first term of the velocity for both vortical and potential regions and denote $\boldsymbol{V}' = \{V_x', V_y', V_z'\} = \boldsymbol{P}\boldsymbol{V}$. The relative velocity $\boldsymbol{P}_t\boldsymbol{X}$ can be omitted as an additive term that is cancelled in the continuity conditions across the vortex boundary irrespective of its shape. In the following let $\boldsymbol{V}_c'$ stand for the set of complex quantities $\{V_1', V_2', V_3'\}$ resulting from the transformation of

V'_x, V'_y, V'_z by analogy with (4.5): $\boldsymbol{V}'_c = \boldsymbol{T}\,\boldsymbol{V}'$ (note that $V'_z \equiv V'_3$). The linear relationship between $\boldsymbol{V}'_c$ and $\boldsymbol{\xi}$ follows from the introduced notation and (4.6): $\boldsymbol{V}'_c = \boldsymbol{TPT}^*\boldsymbol{L}_t\,\boldsymbol{\xi}$. Using (4.11) to eliminate the Lagrangian variables we obtain the Eulerian expression for the complex velocity:

$$\boldsymbol{V}'_c = \boldsymbol{E}\,\boldsymbol{W}', \quad \text{where} \quad \boldsymbol{E} = \boldsymbol{TPT}^*\boldsymbol{L}_t\boldsymbol{L}^{-1}\boldsymbol{TP}^{\mathrm{T}}\boldsymbol{T}^*. \tag{4.14}$$

An extensive matrix algebra for the elements of $\boldsymbol{E}$ yields

$$\left.\begin{aligned} E_{11} &= \bar{E}_{22} = -\mathrm{i}\,\frac{\omega\sigma^2 + (\omega-\gamma)\alpha^2 + \omega + \gamma}{\varkappa(\alpha-1)(\alpha+1-2\sigma\beta)}, \\ E_{12} &= \bar{E}_{21} = -\mathrm{i}\varkappa\,\frac{\left[2\omega\sigma^2 + (\omega-4\gamma)\alpha^2 + 6\omega\alpha + 4\gamma + \omega\right]\beta^2 + 2\omega\alpha}{(\alpha-1)(\alpha+1-2\sigma\beta)}, \\ E_{13} &= \bar{E}_{23} = \bar{E}_{31} = E_{32} = -\mathrm{i}\varkappa\gamma\beta, \quad E_{33} = 0, \end{aligned}\right\} \tag{4.15}$$

and for the components of $\boldsymbol{V}'_c$ inside the vortex we have

$$V'_1 = E_{11}\,W'_1 + E_{12}\,W'_2 + E_{13}\,Z', \quad V'_2 = \bar{V}'_1, \tag{4.16a,b}$$

$$V'_z = E_{31}\,W'_1 + E_{32}\,W'_2. \tag{4.16c}$$

This velocity field is three-dimensional but can be represented as a sum of two planar solenoidal vector fields of different orientations: $\boldsymbol{V}'_c = \boldsymbol{V}'^{pot}_c + \boldsymbol{V}'^{rot}_c$, where $\boldsymbol{V}'^{pot}_c$ is potential and $\boldsymbol{V}'^{rot}_c$ is rotational, and has a structure similar to the velocity inside Kirchhoff's elliptical vortex:

$$V'^{pot}_1 = E_{13}\,Z', \qquad V'^{rot}_1 = E_{11}\,W'_1 + E_{12}\,W'_2, \tag{4.17a,b}$$

$$V'^{pot}_z = \bar{E}_{13}\,W'_1 + E_{13}\,W'_2, \qquad V'^{rot}_z = 0. \tag{4.17c,d}$$

Since we omit the relative velocity and simply take projections of the velocity on the instantaneous axes of the moving reference frame, the condition of potentiality has the same form in both stationary and moving coordinates: $\mathrm{rot}_{\boldsymbol{X}}\boldsymbol{V} = \mathrm{rot}_{\boldsymbol{X}'}\boldsymbol{V}' = 0$. In terms of the complex velocity components it is written as

$$\frac{\partial\,V'_1}{\partial\,W'_1} = \frac{\partial\,V'_2}{\partial\,W'_2}, \quad \frac{\partial\,V'_z}{\partial\,W'_2} = \frac{\partial\,V'_1}{\partial\,Z'}. \tag{4.18a,b}$$

The potentiality of $\boldsymbol{V}'^{pot}_c$ given by (4.17*a*), (4.17*c*) is confirmed by direct verification.

We shall consider the potential continuation of two partial velocities $\boldsymbol{V}'^{pot}_c$ and $\boldsymbol{V}'^{rot}_c$ into the outer region independently. Irrespective of the shape of the vortex boundary, the potential component $\boldsymbol{V}'^{pot}_c$ which is linear in coordinates can be extended into the exterior trivially. It suffices to suppose that the interior Eulerian expressions for its components (4.17*a*), (4.17*c*) hold for the outer region. The remaining term $\boldsymbol{V}'^{rot}_c$ is rotational but purely two-dimensional, and the technique of potential continuation for plane motions surveyed in § 3.2 applies to it.

For the parametric continuation of the rotational component $\boldsymbol{V}'^{rot}_c$ (4.17*b*), (4.17*d*) into the exterior, we proceed from the parameterization (4.13*a, b*) of the vortex boundary. The subsequent derivation depends on the ratio of the constants l_1 and l_2 which appear in (4.13*a*). Since we have assumed $\alpha \neq 1$ in (4.1*a*), then $|l_1/l_2| \neq 1$ in conformity with (4.12) and the elliptic cross-section of the vortex described by (4.13*a, b*) does not degenerate into a line. Let us consider the case of $|l_1/l_2| > 1$. Following the method of § 3.2 we obtain the potential velocity

$$V'_1 = (E_{11}\,l_2 + E_{12}\,l_1)\bar{\eta} + (E_{11}\,l_1 + E_{12}\,l_2)/\bar{\eta} \tag{4.19a}$$

at the point of the outer region

$$W_1' = l_1\,\eta + l_2/\eta, \quad |\eta| \geqslant 1, \tag{4.19b}$$

where η is the complex parameter that varies outside the unit circle and becomes $\exp(\mathrm{i}\varphi)$ at the vortex boundary. As $\eta \to \exp(\mathrm{i}\varphi)$, the coordinate W_1' in (4.19*b*) tends to the vortex boundary given by (4.13*a*), and the velocity (4.19*a*) agrees with the rotational part of the interior velocity (4.17*b*), (4.17*d*). It can be shown that the condition $\mathrm{d}W_1'/\mathrm{d}\eta \neq 0$ which is satisfied for (4.19*b*) by virtue of the assumption $|l_1/l_2| > 1$ ensures the absence of branch points of the velocity (4.19*a*) as a function of W_2' in the exterior of the ellipse. In order to obtain an explicit Eulerian expression for velocity we eliminate the parameter η from (4.19*a, b*):

$$\begin{aligned} V_1' = {} & \tfrac{1}{2}(E_{11}\,l_2/l_1 + E_{12})[W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}] \\ & + 2\,l_1\,(E_{11}\,l_1 + E_{12}\,l_2)[W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}]^{-1} \quad \text{for} \quad |l_1/l_2| > 1. \end{aligned} \tag{4.20}$$

Here, the radicals have positive signs since $\bar{\eta} = [W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}]/(2\,l_1)$ must stay outside the unit circle while W_2' tends to infinity. Potentiality of the velocity (4.20) is verified by the condition (4.18*a*) which is satisfied for (4.20) trivially. The main contribution to the asymptotic form of (4.20) far from the vortex (as $|W_2'| \to \infty$) is made by the term linear in the coordinate:

$$V_1' \sim (E_{11}\,l_2/l_1 + E_{12})W_2' + O(W_2'^{\,-1}),$$

while the near circulation field of the vortex decays at infinity in inverse proportion to distance.

The procedure of potential continuation of (4.17*b*), (4.17*d*) for $|l_1/l_2| < 1$ is quite similar except for the choice of the parameterization for the region of potential flow. The problem is solved by setting

$$W_1' = l_1/\bar{\eta} + l_2\bar{\eta}, \quad |\eta| \geqslant 1, \tag{4.21a}$$

$$V_1' = (E_{11}\,l_1 + E_{12}\,l_2)\eta + (E_{11}\,l_2 + E_{12}\,l_1)/\eta, \tag{4.21b}$$

where $\eta = \exp(\mathrm{i}\varphi)$ at the vortex boundary. Since (4.21a) for $|l_1/l_2| < 1$ entails $\mathrm{d}W_1'/\mathrm{d}\bar{\eta} \neq 0$ throughout the outer region, there can be no branching of the velocity (4.21*b*) as a function of the coordinate. The Eulerian expression for velocity takes the form

$$\begin{aligned} V_1' = {} & \tfrac{1}{2}(E_{11}\,l_1/l_2 + E_{12})[W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}] \\ & + 2\,l_2\,(E_{11}\,l_2 + E_{12}\,l_1)[W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}]^{-1} \quad \text{for} \quad |l_1/l_2| < 1. \end{aligned} \tag{4.22}$$

The total three-dimensional velocity outside the vortex is obtained as a sum of $\boldsymbol{V}_c'^{pot}$, which is potential everywhere and given by (4.17*a, c*), and the potential continuation of the component $\boldsymbol{V}_c'^{rot}$, (4.20), (4.22). The resultant expression for any proportion $|l_1/l_2|$ takes the form

$$\left.\begin{aligned} V_1' = {} & \tfrac{1}{2}(E_{11}\,l_{min}/l_{max} + E_{12})[W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}] \\ & + 2\,(E_{11}\,l_{max}^2 + E_{12}\,l_1 l_2)\left[W_2' + (W_2'^2 - 4\,l_1 l_2)^{1/2}\right]^{-1} + E_{13}\,Z', \\ V_z' = {} & \mathrm{Re}\{E_{13}\,W_2'\}, \quad V_2' = \bar{V}_1', \end{aligned}\right\} \tag{4.23}$$

where l_{max} and l_{min} are the largest and the smallest coefficient of l_1, l_2 in absolute value, respectively. As seen from these formulae, the dependence of the velocity components transversal to the vortex axis on the coordinates is nonlinear, but the asymptotic behaviour of the velocity at a large distance from the vortex remains linear in the coordinates:

$$V_1' \sim (E_{11}\, l_{min}/l_{max} + E_{12})W_2' + E_{13}\, Z' + O(W_2'^{-1}),$$
$$V_z' \sim \mathrm{Re}\{E_{13}\, W_2'\},$$

which corresponds to a stationary linear straining flow or a deformation potential quadratic in coordinates X', Y', Z' far from the vortex. As seen from (4.15), one of the coefficients of the external strain, E_{13}, never vanishes for non-degenerate precession of the vortex when both λ and γ are not zero. Consequently, the freely precessing vortex (4.1*a–c*) (i.e. not connected with an exterior strain flow) in the form of an elliptic cylinder cannot exist. It is now clear that the three-dimensional precession of the cylindrical vortex may occur only under the action of a coordinated strain that changes its orientation in accordance with the motion of the vortex and is non-stationary and periodical as a function of fixed coordinates X, Y, Z.

In spite of the fact that the motion of the vortex is non-autonomous, the mechanism of forced precession studied above may represent a non-trivial example of the interaction between three-dimensional vortex structures and ambient flows.

There are also several noteworthy two-dimensional special cases of the solution derived in this section. First, for $\gamma = 0$ in (4.1*a–c*) our model reduces to the stationary two-dimensional columnar vortices of elliptic cross-section in a plane strain studied by Moore & Saffman (1971). They showed, however, that certain two-dimensional configurations of this type are unstable if the strain is strong enough and the aspect ratio of the elliptic cross-section is large enough. Apart from that, setting $\lambda = \beta = 0$ in (4.1*a–c*) turns the precession of the moving reference frame X', Y', Z' into rotation about the Z-axis according to (4.8*a, b*), (4.9) and reduces the solutions (4.14), (4.23) to a variety of two-dimensional vortices in a rotating fluid. They appear to be quite analogous to the steady states of two-dimensional elliptic vortices in externally imposed strain combined with rotation of constant rate γ, that were described in Kida (1981) (see also Saffman 1992, § 9.3).

Our study of three-dimensional precessing vortices is restricted to the rotational flow (4.1*a–c*) that belongs to the more general set of solutions discussed in § 3. But the method developed for (4.1*a–c*) should work as well for the other types of solutions in the case of rectilinear vortex lines. In particular, the results that we obtained above apply to the solutions (3.26*a, b*) for a linear function $h(c) = \beta c$:

$$W_1 = \left[(2 - \omega/\gamma)\xi_1\, \mathrm{e}^{-\mathrm{i}\omega t} + (2 + \omega/\gamma)\bar{\xi}_1\, \mathrm{e}^{\mathrm{i}\omega t} + \beta c\right] \mathrm{e}^{-\mathrm{i}\gamma t}, \quad Z = c.$$

Solutions of this type do not require a separate study, since the above expression for particle trajectories is at the same time the particular case of (4.1*a–c*) for $\alpha = (2\gamma + \omega)/(2\gamma - \omega)$ (up to an appropriate scaling of the Lagrangian variables).

The problem of isolated vortex tubes may be formulated on the basis of the solutions of § 3 much more generally. Worthy of consideration, in our opinion, are vortices of a non-elliptic transverse section. It is also interesting to determine possible shapes of precessing curved vortex filaments of finite thickness using the solution (3.26*a, b*) with a nonlinear function $h(c)$. However, such problems require significant extension of the available techniques and are beyond the scope of this paper.

5. Concluding remarks

This paper develops the concept of fluid motion as a continuous deformation of infinitesimal material vector elements. The Jacobi matrix that obeys the governing equations (2.8)–(2.10) is shown to be the most adequate and natural characteristic of motion for this concept. The matrix ('deformation') approach is distinct from the conventional Lagrangian ('trajectory') formulation and is, actually, similar to the tensor description in the elasticity theory.

Most of the potential advantages of the proposed approach are yet to be explored, so we confine ourselves to just a few remarks. To illustrate the potentialities of the matrix formulation we derived from the system of matrix equations several new classes of non-stationary three-dimensional rotational flows that would be extremely difficult to find within the framework of conventional formulations. For instance, the solution (3.30*a*–*c*) written for real X, Y, Z and a, b, c satisfies the original Lagrangian equations (2.2), (2.3) but becomes so cumbersome and complicated that it is difficult to find from (2.2), (2.3) directly (note that the Eulerian expression for the velocity field corresponding to (3.30*a*–*c*) cannot be obtained in an explicit form at all).

The matrix approach admits a flexible formulation of the problem of interest and offers a compact 'block' representation of non-trivial three-dimensional motions (e.g. (3.6) embraces not only the entire family of flows with precessing vorticity but also the plane Ptolemaic flows). It allows one to take advantage of the powerful machinery of matrix calculus, which is especially important for numerical simulations. These features may also be useful for developing perturbation methods for a wide class of particular problems. It is also worth noting that, surprisingly, the matrix formulation makes it possible to study three-dimensional rotational motions using analytic functions of complex variables, while traditionally the field of application of analytic functions is confined to plane potential motions.

In our opinion, the set of matrix equations studied in this paper does not exhaust the potential of the Lagrangian matrix approach. Along with new applications, extension of the theory taking into account viscous effects, non-uniform density, and compressibility, is an interesting and, we believe, realistic prospect.

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